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EXACT EVALUATION OF AN ULTRASONIC SCATTERING FORMULA FOR A RIGID IMMOVABLE SPHERE

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Abstract—It has been discovered that three theoretically derived plots of acoustic scattering for rigid immovable spheres appearing in two standard reference works are only qualitatively correct and therefore could be misleading. The reason for the inaccuracy is shown to be premature truncation of the infinite series resulting from the theory. Quantitative plots resulting from exact computations and those corresponding to premature truncations are displayed. The main purpose of this note is to alert others, who may wish to check their programming of theoretical expressions for scattering of ultrasonic waves from spheres, to the error of using the plots in the standard reference works and to suggest that they use the exact results presented here.

In pursuing a theoretical description of backscattering characteristics of the lung surface we employed a theoretical expression, the derivation of which appears in *Theoretical Acoustics* by Morse and Ingard (1968). This expression describes scattering of acoustic plane waves of unit amplitude from a compressible sphere of finite density. The expression for the pressure variation, $p_s(r, \theta)$, before taking the real part, is

$$p_s(r,\theta) = \sum_{m=0}^{\infty} (2m+1) i^m P_m(\cos(\theta)) \times \frac{j_m'(ka) + i\beta_m j_m(ka)}{h_m'(ka) + i\beta_m h_m(ka)} h_m(kr) \qquad (1)$$

where

$$\beta_m = i \frac{\rho c}{\rho_e c_e} \frac{j_m'(k_e a)}{j_m(k_e a)} \quad \text{and} \quad i = \sqrt{-1}.$$

In this expression r is the distance from the center of the scattering sphere to the field point, θ is the scattering angle, and a is the radius of the scattering sphere. k, ρ and c, are respectively, the wave number, the density, and the speed of sound for the medium surrounding the scattering sphere; k_e, ρ_e, and c_e are the corresponding quantities for the material of which the scattering sphere is composed. P_m is the Legendre polynomial of order m, j_m is the spherical Bessel function of the first kind and order m, and h_m is the spherical Hankel function of the first kind and order m.

$$j_m'(x) \equiv \frac{dj_m(x)}{dx} \quad \text{and} \quad h_m'(x) \equiv \frac{dh_m(x)}{dx}.$$

Equation (1) was evaluated using FORTRAN-IV-PLUS on a PDP 11/34 computer.

To test the validity of the program, tabulated or graphed evaluations of expression (1) for sample values of k, a, r, ρ, c, ρ_e and c_e as functions of angle (θ) were sought. The only such evaluations which were found appear in the form of plots of intensity vs angle in *Vibration and Sound* by Morse (1948). These plots were reproduced in subsequent didactic publications (Morse *et al.*, 1968; Nicholas, 1977). The plots are done for rigid immovable spheres corresponding to the conditions $\rho_e \to \infty$ and $\kappa_e \to 0$ in equation 1; these conditions can be shown to imply that $\beta_m \to 0$ for all m. The plots of intensity vs angle are done for three cases:

$$ka = 1, \quad ka = 3, \quad \text{and} \quad ka = 5.$$

The relation of r to a was not stated in the above references.

Setting $\beta_m = 0$ (for all m) and $r \gg a$ in our program, plots of scattered intensity vs angle were generated for the same three cases ($ka = 5, 3, 1$). These are shown in Fig. 1. In generating these plots, it was found necessary to sum as many as twelve terms in the series (equation 1) for convergence. A significant lack of agreement was apparent to us between our results, illustrated in these figures, and those displayed in Morse (1948). Since

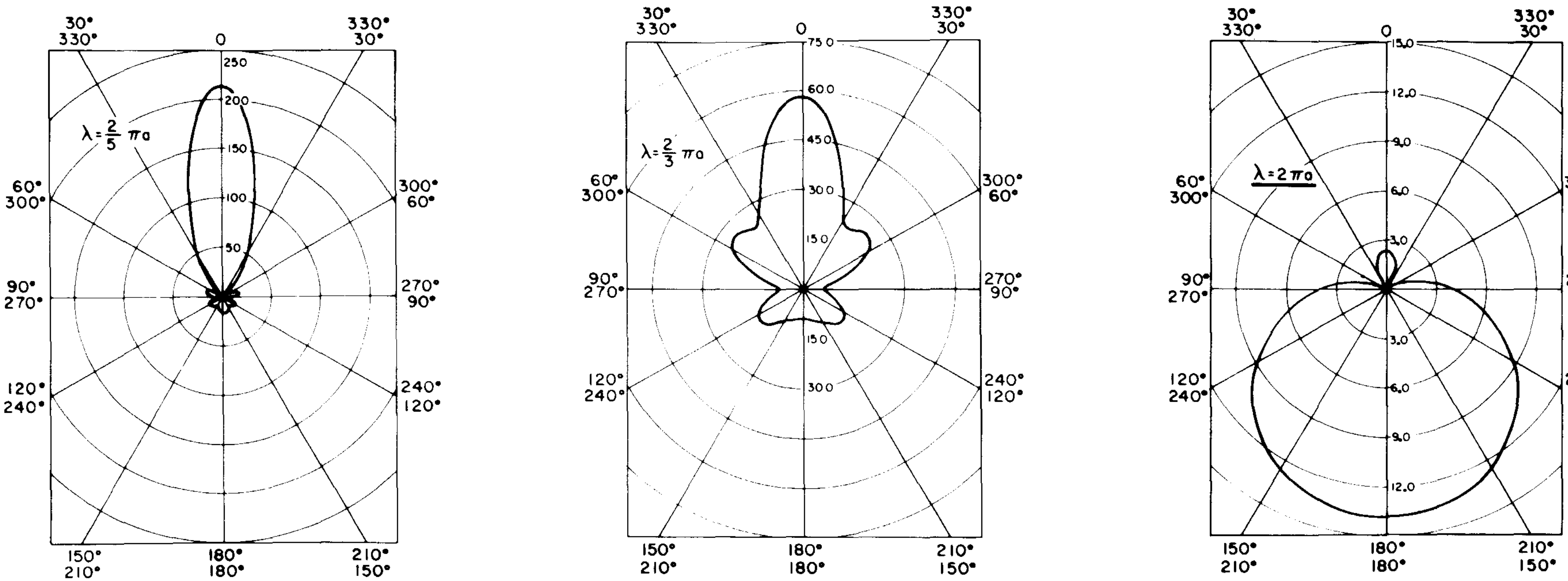

Fig. 1. Distribution in angle of $p_s p_s^* \times 10^6$ (which is proportional to the scattered intensity) from a rigid immovable sphere for $\lambda = 2/5\pi a$, $2/3\pi a$ and $2\pi a$ ($ka = 5, 3$ and 1 respectively). a is the radius of the sphere and λ is the wavelength. The plots are done for $r \gg a$, and r/a is the same for the three cases, where r is the distance from the center of the sphere to the field point.

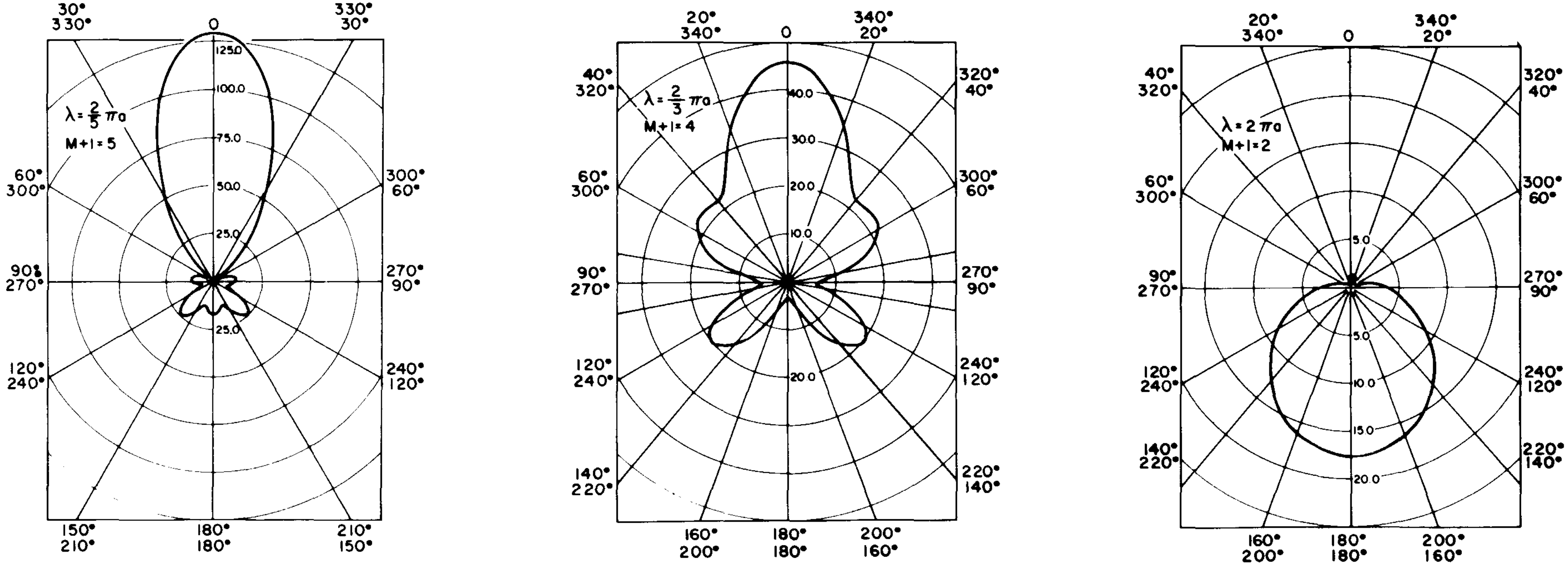

Fig. 2. Plots of $p_s p_s^* \times 10^6$ (which is proportional to the scattered intensity) vs angle for the same three cases as in Fig. 1, but generated with premature truncation of the series in equation 1. The number of terms employed (indicated by $M + 1$ in each case) was chosen to give agreement with the results of Morse (1948) for the same λ/a ratios. Exact agreement was obtained for $ka = 5$ and 3, while approximate agreement was obtained for $ka = 1$. As in Fig. 1, $r \gg a$ and r/a is the same for all cases.

the values of r/a were not stated in Morse regarding his plots, it was decided to test our results relaxing the restriction that $r \gg a$. The intensity distribution for the three scatter sizes was investigated over the entire range of r/a, i.e. $1 < r/a < \infty$. Worse agreement with Morse's plots occured for $r \not\gg a$ than for $r \gg a$. Extensive searches for programming errors produced no changes in the computed results.

Equation 1 was also evaluated using a programmable calculator (TI 59) for the three cases with $\beta_m = 0$ and $r \gg a$. The results agreed exactly with our PDP 11/34 results. Chances for identical programming errors were minimal since both programs were developed independently by separate programmers using different languages.

It was then suspected that the disagreement between our results and Morse's was due to premature truncation of the series in equation 1 on the part of those who generated the plots in Morse. We then generated plots for the three cases keeping the first $M + 1$ terms in the series for $M + 1 = 1, 2, 3, \ldots$. It was found that, for $r \gg a$ and $M + 1 = 5$ and 4, exact replicas of Morse's curves for $ka = 5$ and 3, respectively, were obtained. Thus, premature truncation must indeed exist; our truncated results are displayed in Fig. 2. For the case of $ka = 1$ an exact replica of Morse's curve was not obtained for any choice of $M + 1$ (maintaining $r \gg a$); reasonably good agreement was obtained, however, for the choice of $M + 1 = 2$.

The results of this analysis are significant in that both the shapes of the curves and numerical values in Figs 1 and 2 differ considerably. The plots given by Morse might serve as rough guides regarding the angular dependence of scattering on scatterer diameter. However, any quantitative use of Morse's plots, such as testing the programming of equation 1, is not valid.

The results displayed in Fig. 1 are rigorous and provide accurate quantitative information. These plots of $p_s p_s^* \times 10^6$ (proportional to the scattered intensity) vs angle result from exact evaluation of equation 1 corresponding to a rigid immovable sphere with the condition $r \gg a$. The ratio r/a, chosen as 125.7, is the same for all plots; therefore, a comparison of the intensity scattered for particles of diameters $a = 5/k$, $3/k$ and $1/k$ can be made, the incident plane wave having the same amplitude and wavelength in all three cases.

A more rigorous expression than Morse's for scattering from spheres was derived by Faran (1951), in that existence of shear waves inside the sphere is allowed. For a rigid immovable sphere, however, the scattered pressure amplitude given by Faran can also be shown to reduce to equation 1 with $\beta_m = 0$ (for all m). Thus the plots in Fig. 1 apply to Faran's results (for a rigid immovable sphere) as well as to Morse's.

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