Method of data reduction for accurate determination of acoustic backscatter coefficients

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(Received 2 October 1983; accepted for publication 29 May 1984)

In previous methods of data reduction used to determine ultrasonic backscatter coefficients, various approximations were made. One frequently used is that there is an abrupt cutoff in the lateral extent of the scattering volume interrogated. Another approximation in all previous methods is that the effect of time gating the received echo signals can be written as a function of the distance along the axis of the interrogating beam. In the present paper we show that the backscatter coefficient can be derived from experimental data without making such approximations. The cases of narrow-band and broadband pulses are treated, and the method is applicable whatever the distance between the interrogated volume of scatterers and the transducer face. It is shown that, for a given pulse form, the gate duration must be sufficiently long in order to attain a specified accuracy for the measured backscatter coefficient. A test of the method was done using a phantom with well-defined scattering properties. Very good agreement was found between measured values of backscatter coefficients and those calculated using a first-principles theory.

PACS numbers: 43.20.Fn, 43.20.Bi, 43.60.Gk

INTRODUCTION

The backscatter coefficient $\eta(\omega)$ at the angular frequency ω is defined in ultrasound measurements as the differential scattering cross section per unit volume for a scattering angle of 180°.[1] Ideally, to determine $\eta(\omega)$, relevant measurements would be made in which plane sinusoidally varying waves were incident on an isolated volume of scatterers and the scattered intensity monitored. If pressure sensitive transducers are used to detect the scattered wave, they should be positioned at a distance far enough from the scattering volume that phase cancellation effects at the receiver can be made negligible. Such an experiment conforms to the definition of differential scattering cross sections, as applied to ultrasound.[2]

Measurements of this type have been reported[3,4] and yield accurate determinations of $\eta(\omega)$. Such measurement techniques are not applicable to *in vivo* situations, however, because the scattering volume must be small (a few cm^3) and surrounded by a nonscattering medium. Thus, *in vivo* measurements must be made under circumstances far removed from the ideal ones just described.

All measurements of $\eta(\omega)$ reported to date, which are applicable *in vivo*, involve the projection of a pulsed beam into a volume of tissue and the monitoring of echo signals due to scattering from a region of the tissue defined laterally by the beam pattern and axially by the pulse duration and the time gate duration. The geometries involved are complicated; e.g., a specific volume of scatterers is not simply defined. Also plane waves are not approximated except over very small volumes (perhaps of the order of 1 mm^3), and large-scale phase cancellation occurs during echo reception at the face of the transducer.

Various approximations have been applied to obtain values for $\eta(\omega)$. For example, some investigators[1,5] have taken the volume of scatterers involved to be bounded laterally at the 3-dB beamwidth in the gated region. Others have chosen the 6-dB beamwidth.[6,7] The calculated backscatter coefficients can differ among these by a factor of 2 just on the basis of the somewhat arbitrary choice of 3 or 6 dB. In later articles by Nicholas,[8] Lizzi *et al.*,[9] and Campbell and Waag,[10] this arbitrariness appears to have been removed, however.

Another approximation common to current methods is that, corresponding to the echo signal at the time t, there exist two planar surfaces perpendicular to the axis of the transducer such that each scatterer lying between these planes contributes to the echo signal and all other scatterers (not lying between these planes) do not contribute. All previous data reduction schemes in the literature make this assumption. Consideration of scatterers far enough from the beam axis should convince the reader that this is not really true. Two such bounding surfaces will exist, but they are not planes; each surface likely has a shape lying between a plane and a spherical surface, the latter having its center at the center of the transducer face.

Because the gate function and the echo wave form are multiplied together, we can expect that the Fourier transform of the gated echo wave form will have contributions from a continuous band of frequencies involving both the pulse emitted and the gate function. (The two functions are convolved.) The presence of this convolution has been recognized by Chivers and Hill.[11] This situation is emphasized in importance, the broader the frequency bands of the pulse and gate.

In this article, we describe a method of data reduction which avoids the uncertainties described above. The method described here will produce a more accurate determination of the backscatter coefficient than the previous methods. The method can be applied for any combination of focused or nonfocused and narrow or broadband transducers. In addition, the region interrogated is not limited to the farfield or (in the case of a focused transducer) to the focal region of the transducer.

An extensive test of this method of data reduction, involving phantoms with well-defined scattering properties, is the subject of a paper in preparation. In the present paper the results of one of these tests is included.

I. THE BACKSCATTER EXPERIMENT AND NATURE OF THE RAW DATA

The experiment used for determining backscatter coefficients will be described using Figs. 1 and 2. A single transducer is used to generate a pulse and detect subsequent echoes from the scattering medium. The transducer involved can be focused or nonfocused and pulses created by the transducer are separated enough in time that all echo signals analyzed correspond to a single pulse. Also, all signal amplitudes are low enough—and the electronics are such—that all responses are linear.

In Fig. 1 the transducer and block of scattering material are shown, the beam axis being perpendicular to the planar boundary of the sample's proximal surface. This boundary is chosen to be planar to simplify corrections for attenuation in the scattering medium.

The data collection consists of two parts. First, referring to Fig. 1, a set of gated echo signals, due to scattering from within the scattering volume, are recorded. The onset and termination of the gate remains constant relative to the time of emission of the pulse. Between recordings of gated signals, however, the transducer is translated perpendicularly to the beam axis so that the positions of the scatterers contributing to any one gated signal can be considered to be uncorrelated spatially with the positions of those contributing to any of the other gated signals recorded. Also, the number of such gated echo signals recorded must be sufficiently large that mean values are significant; e.g., 25 such recorded signals would generally suffice.

The final piece of data consists of a recording of the echo signal from a reference reflector. An example of a reference reflector is shown in Fig. 2. This consists of a planar specular reflector placed perpendicular to the beam axis. The reflector is assumed to be planar and specular in the method of data reduction described below. The method could, however, be modified to accommodate any reference reflector having a sufficiently well-known geometry. In the technique of Sigelmann and Reid a planar reflector is employed to estimate "the power reflected from a known interface."[1] In our technique the entire time-dependent echo signal is recorded and used in the determination of the backscatter coefficient.

II. THE METHOD OF DATA REDUCTION YIELDING THE BACKSCATTER COEFFICIENT

A. Assumptions involving the scattering medium

Three assumptions are made about the scattering medium. First of all, the wave fronts of the scattered wave from each scatterer involved are approximated to be spherical in the region of the transducer face; this requires that the scatterer either be monopolar in nature or be sufficiently far from the transducer face. Thus, if the scatterers are not well-defined, as in tissues, the experiment should involve a sufficiently large distance between the transducer face and the volume of scatterers interrogated.

Second, we assume that all scatterers are discrete and identical in this derivation. The method can easily be generalized, however, to apply to a finite number of sets of scat-

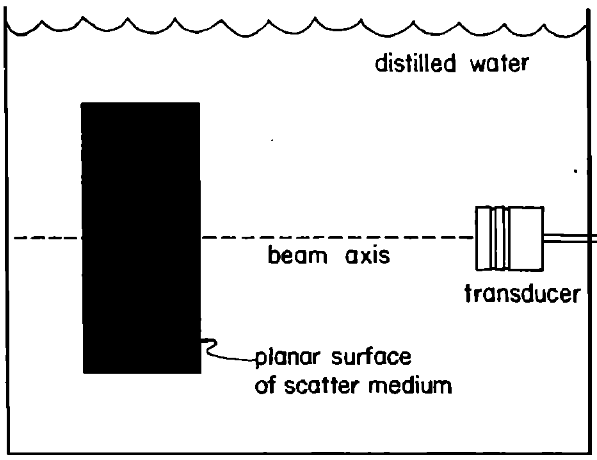

FIG. 1. Depiction of the experimental situation for collecting the echo signals backscattered from the scattering medium. The scattering medium is in the form of a cylinder with its plane parallel surfaces perpendicular to the beam axis.

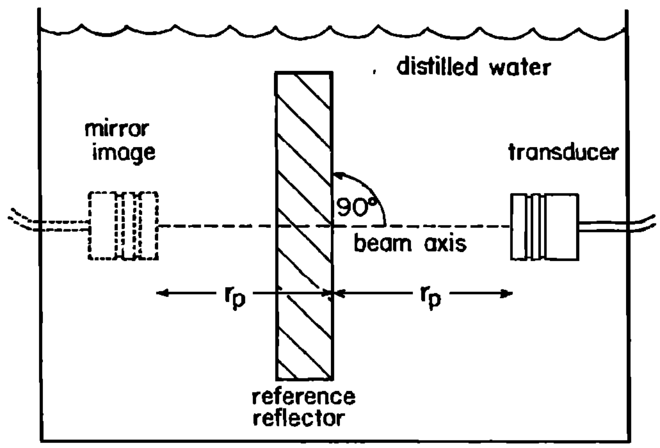

FIG. 2. Depiction of the experimental situation for collecting the reference echo signal from the planar reflector. The "mirror image" receiving transducer is shown on the left at a distance of $2r_p$ from the actual interrogating transducer.

terers, each set containing scatterers identical to one another but unlike those in other sets.

Third, it is assumed that the scatterers are randomly distributed in space and that the average number per unit volume is small enough that the only (apparent) coherent scattering is related to the onset and termination of the gate. All other scattering is incoherent. This idea is expanded upon in the Appendix.

B. Derivation of the method and its application to the case of narrow banded pulses

Define a pressure "wave packet" produced by the transducer with $p(\mathbf{r},t)$. This is the instantaneous pressure at the head of the position vector $\mathbf{r}$ at the time t. We can write this as the superposition of a complete set of continuous wave beams varying sinusoidally in time at any point in space.[12]

$$p(\mathbf{r},t) = \mathrm{Re}\int_0^\infty d\omega\, A(\mathbf{r},\omega)e^{-i\omega t}$$

$$= \iint_S ds'\, \underbrace{\mathrm{Re}\int_0^\infty d\omega\, A_{00}(\omega)\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|-i\omega t)}{|\mathbf{r}-\mathbf{r}'|}}_{\substack{\text{pressure at the field point } \mathbf{r} \text{ and time } t \text{ due to}\\ \text{pulsed monopole radiators in the area element}\\ ds' \text{ on the face of the transducer}}},$$

where the beam is assumed to be formed from a superposition of pressure waves emitted in unison from monopole radiators uniformly distributed over the transducer face S. The quantity $A_{00}(\omega)$ is a complex superposition coefficient, and Re means "real part of the quantity following." The complex wavenumber is denoted by $k = \omega/c(\omega) + i\alpha(\omega)$, where $c(\omega)$ is the speed of sound and $\alpha(\omega)$ is the attenuation coefficient at the angular frequency ω.

We can also write $p(\mathbf{r},t)$ in the form

$$p(\mathbf{r},t) = \mathrm{Re}\int_0^\infty d\omega\, A_{00}(\omega)\iint_S ds'\,\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|-i\omega t)}{|\mathbf{r}-\mathbf{r}'|}$$

$$= \mathrm{Re}\int_0^\infty d\omega\, A_{00}(\omega)e^{-i\omega t}A_0(\mathbf{r},\omega),$$

where

$$A_0(\mathbf{r},\omega) \equiv \iint_S ds'\,\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}$$

is proportional to the Rayleigh integral[13] for the case in which the normal component of the velocity at any instant of time is the same at all points on the radiating surface. Negative frequencies, though redundant, can be conveniently introduced. Let $c(\omega) = c(-\omega) = |c(\omega)|$ and $\alpha(\omega) = \alpha(-\omega) = |\alpha(\omega)|$. Then, since $e^{-i\omega t}A_0(\mathbf{r},\omega)$ is a solution of the wave equation, so is its complex conjugate; however, its complex conjugate also equals $e^{i\omega t}A_0(\mathbf{r},-\omega)$. Requiring $A_{00}(-\omega) = A^*_{00}(\omega)$ where the asterisk denotes complex conjugation, we have

$$P(\mathbf{r},t) = \frac{1}{2}\int_{-\infty}^\infty d\omega\, A_{00}(\omega)e^{-i\omega t}A_0(\mathbf{r},\omega).$$

Define $B_0(\omega) \equiv (1/2)A_{00}(\omega)$. Then we have the simple form

$$p(\mathbf{r},t) = \int_{-\infty}^\infty d\omega\, B_0(\omega)e^{-i\omega t}A_0(\mathbf{r},\omega),$$

where $B_0(\omega)$ is a complex superposition coefficient. Notice that $A_0(\mathbf{r},\omega)$ depends only on ω, $c(\omega)$, $\alpha(\omega)$, and the shape and size of the transducer piezoelectric element.

Now suppose a scatterer exists at the position $\mathbf{r}$. Assuming that the scatterer is small enough (e.g., less than a millimeter in diameter), we can approximate the incident wave packet as a superposition of plane waves each having a sinusoidal time dependence. In fact, we can deal only with the plane wave having frequency ω, i.e., that component having frequencies between ω and $\omega + d\omega$ where $d\omega$ is a differential. We have

$$dp_\omega(\mathbf{r},t) \equiv d\omega\, B_0(\omega)e^{-i\omega t}A_0(\mathbf{r},\omega). \quad (1)$$

The pressure wave scattered is given by

$$dp_{\omega s}(\mathbf{r}',t) = dp_\omega(\mathbf{r},t)\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\Phi(k,\cos\theta),$$

where we have assumed that $|\mathbf{r}-\mathbf{r}'| \gg 1$ mm.[14] θ is the scattering angle between the direction of propagation of the incident plane wave and $\mathbf{r}'-\mathbf{r}$. $\mathbf{r}'$ is the point in space at which the scattered pressure is $dp_{\omega s}(\mathbf{r}',t)$ at the time t.

The force on the transducer face at time t is found by integrating over the area of the transducer face.

$$dF_s(\mathbf{r},\omega,t) = dp_\omega(\mathbf{r},t)\Phi(k)\iint_S ds'\,\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}.$$

The area element ds' is at the head of $\mathbf{r}'$. We have invoked our assumption that the scatterer is monopolar in nature or that the scatterer is far enough from the transducer face that we can take

$$\Phi(k,\cos\theta) \approx \Phi(k,\cos 180^\circ) \equiv \Phi(k);$$

i.e., $\Phi(k)$ has no θ dependence. Notice that the area integral has already been defined as $A_0(\mathbf{r},\omega)$. Thus, we have

$$dF_s(\mathbf{r},\omega,t) = d\omega\, B_0(\omega)e^{-i\omega t}[A_0(\mathbf{r},\omega)]^2\Phi(k).$$

It can be shown[15] that

$$\Phi(k) \equiv \psi(\omega) = \psi^*(-\omega). \quad (2)$$

Also, $\|\Phi(k)\|^2 = \|\psi(\omega)\|^2$ is the differential scattering cross section of the scatterer at 180°. In this discussion we have referred to the scattering theory for spherical scatterers.

The above discussion applies to a single scatterer at position $\mathbf{r}$. The scattering medium will possess many scatterers, however, and we can define $N(\mathbf{r})$ to be the number of scatterers per unit volume at $\mathbf{r}$. $N(\mathbf{r})$ could, for example, be a sum of delta functions

$$N(\mathbf{r}) = \sum_{i=1}^{M}\delta(\mathbf{r}_i - \mathbf{r}),$$

where the ith scatterer is at position $\mathbf{r}_i$. Then the force at time t on the transducer face due to frequencies between ω and $\omega + d\omega$ can be written

$$dF_s(\omega,t) = d\omega\, B_0(\omega)e^{-i\omega t}\psi(\omega)\iiint_\Omega d\mathbf{r}\, N(\mathbf{r})[A_0(\mathbf{r},\omega)]^2,$$

where Ω is a volume containing all scatterers in the field.

The total force at time t on the transducer face due to all scatterers and including all frequencies is

$$F_s(t) = \int_{-\infty}^{\infty} d\omega\, B_0(\omega) e^{-i\omega t} \psi(\omega) \times \iiint_{\Omega} d\mathbf{r}\, N(\mathbf{r}) [A_0(\mathbf{r},\omega)]^2 .$$

Because the force is computed at some specific time t, not all scatterers contribute; e.g., if t is short enough, $F_s(t) = 0$ because the scattering volume is assumed not to be in contact with the transducer.

Let $T(\omega)$ be the (frequency-dependent) complex receiving transfer function of the transducer. Then the echo signal voltage has the form

$$V_s(t) = \int_{-\infty}^{\infty} d\omega\, T(\omega) B_0(\omega) e^{-i\omega t} \psi(\omega) \times \iiint_{\Omega} d\mathbf{r}\, N(\mathbf{r}) [A_0(\mathbf{r},\omega)]^2 . \quad (3)$$

$V_s(t)$ is the echo signal voltage which will be time gated and then Fourier analyzed. Before doing that, however, let us consider the echo voltage due to the presence of a reference reflector, the scattering medium being removed. Knowledge of the latter allows determination of the factor $T(\omega)B_0(\omega)$ in Eq. (3).

Recall from Sec. I that the pulses created for the scattering situation are identical to those for the reference reflector situation. Thus, at some point $\mathbf{r}$, we have the pressure at time t due to frequencies between ω and $\omega + d\omega$, viz.,

$$dp_\omega(\mathbf{r},t) = d\omega\, B_0(\omega) e^{-i\omega t} A_0(\mathbf{r},\omega) . \quad (1')$$

For simplicity of discussion, let the reflector be planar and perpendicular to the beam axis (as in Fig. 2). If the distance from the center of the transducer face to the plane is r_p, then the force on the transducer face will be given by

$$dF_r(r_p,\omega,t) \equiv d\omega\, B_0(\omega) e^{-i\omega t} R \iint_{S_{\rm mir}} ds\, A_0(\mathbf{r},\omega) .$$

The integral is over the area $S_{\rm mir}$ of a "receiving" transducer face which corresponds to the mirror image of the transmitting transducer in the reflecting plane. The distance between source and receiver is $2r_p$. R is the amplitude reflection coefficient for the planar reflecting surface. The differential area element ds lies at the head of $\mathbf{r}$.

Introducing the complex receiving transfer function and integrating over all ω, we have the echo signal voltage at time t due to the presence of the reference reflector:

$$V_r(t) \equiv \int_{-\infty}^{\infty} d\omega'\, T(\omega') B_0(\omega') e^{-i\omega' t} R \iint_{S_{\rm mir}} ds\, A_0(\mathbf{r},\omega') .$$

For didactic convenience, we have changed the dummy integration variable from ω to ω'. Multiplying by $(1/2\pi)e^{i\omega t}$ and integrating over all time, we have the Fourier transform of the echo voltage:

$$\widetilde{V}_r(\omega) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, V_r(t) e^{i\omega t} = T(\omega) B_0(\omega) R \iint_{S_{\rm mir}} ds\, A_0(\mathbf{r},\omega) . \quad (4)$$

Notice that R should be calculable if the materials have known acoustic properties, $V_r(t)$ is recorded directly in the experiment, and the quantity

$$\iint_{S_{\rm mir}} ds\, A_0(\mathbf{r},\omega)$$

can be calculated, e.g., using a single integration for focused or nonfocused transducers.[12] Therefore, Eq. (4) can be solved for $T(\omega)B_0(\omega)$ for any angular frequency ω. Thus, returning to Eq. (4), $T(\omega)B_0(\omega)$ can be considered known and is given by

$$T(\omega) B_0(\omega) = \widetilde{V}_r(\omega) \left(R \iint_{S_{\rm mir}} ds\, A_0(\mathbf{r},\omega) \right)^{-1} .$$

The next step in our data reduction scheme is to take the Fourier transform of the set of gated echo voltages. Assuming a simple rectangular gate function of the form

$$G(t) = \begin{cases} 1, & \text{if } T_1 \leqslant t \leqslant T_2 , \\ 0, & \text{for all other } t , \end{cases}$$

then

$$\widetilde{V}_s(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, G(t) e^{i\omega t} \int_{-\infty}^{\infty} d\omega'\, T(\omega') B_0(\omega') e^{-i\omega' t} \psi(\omega') \iiint_{\Omega} d\mathbf{r}'\, N(\mathbf{r}') [A_0(\mathbf{r}',\omega')]^2$$

$$= \frac{1}{2\pi} \int_{T_1}^{T_2} dt\, e^{i\omega t} \int_{-\infty}^{\infty} d\omega'\, T(\omega') B_0(\omega') e^{-i\omega' t} \psi(\omega') \iiint_{\Omega} d\mathbf{r}'\, N(\mathbf{r}') [A_0(\mathbf{r}',\omega')]^2$$

and

$$\widetilde{V}_s^*(\omega) \widetilde{V}_s(\omega) = (2\pi)^{-2} \int_{-\infty}^{\infty} d\omega'\, T^*(\omega') B_0^*(\omega') \psi^*(\omega') \int_{-(\tau/2)}^{\tau/2} dt'\, e^{-i(\omega - \omega')t'} \times \int_{-\infty}^{\infty} d\omega''\, T(\omega'') B_0(\omega'') \psi(\omega'') \int_{-(\tau/2)}^{\tau/2} dt''\, e^{i(\omega - \omega'')t''}$$

$$\times \iiint_{\Omega} d\mathbf{r}' \iiint_{\Omega} d\mathbf{r}''\, N(\mathbf{r}')N(\mathbf{r}'')[A_0^*(\mathbf{r}',\omega')]^2[A_0(\mathbf{r}'',\omega'')]^2 , \tag{5}$$

where the zero of the clock has been set in the middle of the time gate so that $T_2 = -T_1 \equiv \tau/2$, and the gate duration is τ. Note also the fact that

$$\int_{-(\tau/2)}^{\tau/2} dt\, e^{i(\omega-\omega')t} = \tau \frac{\sin[(\omega-\omega')(\tau/2)]}{(\omega-\omega')(\tau/2)} \equiv \tau\, \mathrm{sinc}[(\omega-\omega')(\tau/2\pi)] .$$

Recall that in the experiment a set of many gated echo signals is recorded such that each signal corresponds to a distribution of scatterers which is not spatially correlated with that corresponding to any other recorded signal in the set. Expression (5) is obtained by Fourier analysis of one of these echo signals.

Now we take the average of all such quantities. Ideally, an infinite number of measurements would contribute to this average; this would correspond to the ensemble average for all equally likely spatial arrangements of the particles. In practice, of course, a finite number of measurements of $V_s(t)$ contributes. We designate the averaged quantities with a bar over them. We have

$$\overline{\tilde{V}_s^*(\omega)\tilde{V}_s(\omega)} = \left(\frac{\tau}{2\pi}\right)^2 \int_{-\infty}^{\infty} d\omega' T^*(\omega')B_0^*(\omega')\psi^*(\omega')\mathrm{sinc}[(\omega-\omega')(\tau/2\pi)] \int_{-\infty}^{\infty} d\omega''\, T(\omega'')B_0(\omega'')\psi(\omega'')$$
$$\times \mathrm{sinc}[(\omega-\omega'')(\tau/2\pi)]\, \overline{\iiint_{\Omega} d\mathbf{r}' \iiint_{\Omega} d\mathbf{r}''\, N(\mathbf{r}')N(\mathbf{r}'')[A_0^*(\mathbf{r}',\omega')]^2[A_0(\mathbf{r}'',\omega'')]^2} . \tag{6}$$

The averaging of the spatial integrals involves only variations in $N(\mathbf{r}')N(\mathbf{r}'')$ in the integrand; thus, we can write this average as

$$I \equiv \iiint_{\Omega} d\mathbf{r}' \iiint_{\Omega} d\mathbf{r}''\, \overline{N(\mathbf{r}')N(\mathbf{r}'')} \times [A_0^*(\mathbf{r}',\omega')]^2[A_0(\mathbf{r}'',\omega'')]^2 .$$

We have assumed spatial randomness in the distribution of scatterers; i.e., the position of one scatterer is completely independent of that of any other. However, when $\mathbf{r}' = \mathbf{r}''$, there is complete correlation; thus, we can *not* write

$$\overline{N(\mathbf{r}')N(\mathbf{r}'')} = \overline{N(\mathbf{r}')}\ \overline{N(\mathbf{r}'')} .$$

We can, however, separate I into two terms, one corresponding to uncorrelated (or incoherent) scattering and the other to completely correlated (or coherent) scattering. We write

$$N(\mathbf{r}') = \sum_{i=1}^{M} \delta(\mathbf{r}_i - \mathbf{r}')$$

and

$$N(\mathbf{r}'') = \sum_{j=1}^{M} \delta(\mathbf{r}_j - \mathbf{r}'') ,$$

where the total number of scatterers in Ω is M. Then the integrals are replaced by sums:

$$I = \overline{\sum_{i=1}^{M}\sum_{j=1}^{M} [A_0^*(\mathbf{r}_i,\omega')]^2[A_0(\mathbf{r}_j,\omega'')]^2} ,$$

where the bar still represents the ensemble average. The sums can be segregated into two parts, one in which $i = j$ and the other in which $i \neq j$:

$$I = \overline{\sum_{i=1}^{M} [A_0^*(\mathbf{r}_i,\omega')]^2[A_0(\mathbf{r}_i,\omega'')]^2} + \overline{\sum_{i=1}^{M} [A_0^*(\mathbf{r}_i,\omega')]^2 \sum_{\substack{j=1\\ j\neq i}}^{M} [A_0(\mathbf{r}_j,\omega'')]^2} . \tag{7}$$

In writing Eq. (7), the following theorem was invoked: If X_1 and X_2 are random variables with expectations, then the expectation of their sum exists and is the sum of their expectations.[16]

Consider a specific element of the ensemble contributing to the second average on the right of Eq. (7), viz.,

$$\sum_{i=1}^{M} [A_0^*(\mathbf{r}_i,\omega')]^2 \sum_{\substack{j=1\\ j\neq i}}^{M} [A_0(\mathbf{r}_j,\omega'')]^2 .$$

For a large enough value of M, leaving out any one term of the sum

$$\sum_{j=1}^{M} [A_0(\mathbf{r}_j,\omega'')]^2$$

will have a negligible effect; thus,

$$\sum_{\substack{j=1\\ j\neq i}}^{M} [A_0(\mathbf{r}_j,\omega'')]^2$$

is independent of any particular position $\mathbf{r}_i$, i.e., any particular value of i. This means that the second sum in Eq. (7) is the expectation value of the product of two *independent* random variables. The expectation value of the product of two independent random variables equals the product of the expectation values of these random variables.[16,17] Thus we have

$$\overline{\sum_{i=1}^{M}[A_0^*(\mathbf{r}_i,\omega')]^2\sum_{\substack{j=1\\ j\neq i}}^{M}[A_0(\mathbf{r}_j,\omega'')]^2}$$

$$=\overline{\sum_{i=1}^{M}[A_0^*(\mathbf{r}_i,\omega')]^2}\;\overline{\sum_{j=1}^{M}[A_0(\mathbf{r}_j,\omega'')]^2}\,. \tag{8}$$

Now we can write Eq. (7) in the form

$$I=\overline{\sum_{i=1}^{M}[A_0^*(\mathbf{r}_i,\omega')]^2[A_0(\mathbf{r}_i,\omega'')]^2}+\overline{\sum_{i=1}^{M}[A_0^*(\mathbf{r}_i,\omega')]^2}\;\overline{\sum_{j=1}^{M}[A_0(\mathbf{r}_j,\omega'')]^2}\,. \tag{9}$$

Rewriting I in terms of spatial integrals, we have

$$I=\iiint_\Omega d\mathbf{r}'\ \overline{N(\mathbf{r}')}[A_0^*(\mathbf{r}',\omega')]^2[A_0(\mathbf{r}',\omega'')]^2+\iiint_\Omega d\mathbf{r}'\ \overline{N(\mathbf{r}')}[A_0^*(\mathbf{r}',\omega')]^2\times\iiint_\Omega d\mathbf{r}''\ \overline{N(\mathbf{r}'')}[A_0(\mathbf{r}'',\omega'')]^2\,. \tag{10}$$

But $\overline{N(\mathbf{r}')}=\overline{N(\mathbf{r}'')}\equiv\overline{N}$, where $\overline{N}$ is the constant mean (average) number of particles per unit volume, and we have

$$I=\overline{N}\iiint_\Omega d\mathbf{r}'[A_0^*(\mathbf{r}',\omega')]^2[A_0(\mathbf{r}',\omega'')]^2+\overline{N}^2\iiint_\Omega d\mathbf{r}'[A_0^*(\mathbf{r}',\omega')]^2\times\iiint_\Omega d\mathbf{r}''[A_0(\mathbf{r}'',\omega'')]^2\,. \tag{11}$$

Recall that this applies for a sufficiently large number of scatterers and that their positions in space are not correlated.

A similar result to that in Eq. (11) was derived by Glotov[18] in that two terms, one proportional to $\overline{N}$ and the other proportional to $\overline{N}^2$, result. Various distinctions exist between Glotov's treatment and ours, however. His application was specifically for continuous wave bursts of duration τ from a point source followed by reception at the same point. Also, in Glotov's development no gating phenomenon was treated and no direct or indirect reference was made to the essential random variable theorems equivalent to those in Refs. 16 and 17. It is particularly important to realize that our treatment applies for any pulse shape and any type of source and receiver in which the emission of the pulse involves equivalent point sources (acting in unison) uniformly distributed over the transmitting area.

Referring to Eq. (11), as $\overline{N}$ gets larger the coherent ($\overline{N}^2$) term increases more rapidly than the incoherent ($\overline{N}$) term. An argument is presented in the Appendix that the coherent term, as it arises in the complete expression for $\overline{\tilde{V}_s^*(\omega)\tilde{V}_s(\omega)}$ [Eq. (6)], is related primarily to the distal and proximal surfaces of the volume of scatterers (relative to the transducer), whereas the incoherent term is approximately proportional to the volume of scatterers interrogated.

A subtle effect of the particle density has not been discussed above. If the particle density is large enough, then coherent backscattering is approached in the volume of scatterers (this coherent effect is not related to the gate), and the actual backscatter coefficient drops off with increasing $\overline{N}$. This has been discussed by others[19,20] and relevant measurements made by Shung.[21] Thus, for a high enough concentration of scatterers, the *simple* separation of $\overline{\tilde{V}_s^*(\omega)\tilde{V}_s(\omega)}$ into terms as in Eq. (11) is not valid and one needs to begin with Eq. (6) and deal with the factor $\overline{N(\mathbf{r}')N(\mathbf{r}'')}$ in another way. In the present work, we will restrict ourselves to situations in which $\overline{N}$ is small enough that Eq. (11) is valid.

We have not yet obtained an expression for the backscatter coefficient, $\eta(\omega)$, at angular frequency ω. As mentioned above, $\|\psi(\omega)\|^2\equiv\psi^*(\omega)\psi(\omega)$ is the differential scattering cross section for one scatterer at the frequency ω and a scattering angle of 180°. In pursuing an expression for $\eta(\omega)$, we first write out $\overline{\tilde{V}_s^*(\omega)\tilde{V}_s(\omega)}$ including I as shown in Eq. (11):

$$\overline{\tilde{V}_s^*(\omega)\tilde{V}_s(\omega)}=\left(\frac{\tau}{2\pi}\right)^2\int_{-\infty}^{\infty}d\omega'\ T^*(\omega')B_0^*(\omega')\psi^*(\omega')\mathrm{sinc}[(\omega-\omega')(\tau/2\pi)]\times\int_{-\infty}^{\infty}d\omega''\ T(\omega'')B_0(\omega'')\psi(\omega'')\mathrm{sinc}[(\omega-\omega'')(\tau/2\pi)]$$
$$\times\left(\overline{N}\iiint_\Omega d\mathbf{r}'\ [A_0^*(\mathbf{r}',\omega')]^2[A_0(\mathbf{r}',\omega'')]^2+\overline{N}^2\iiint_\Omega d\mathbf{r}'[A_0^*(\mathbf{r}',\omega')]^2\iiint_\Omega d\mathbf{r}''[A_0(\mathbf{r}'',\omega'')]^2\right).$$

This expression can be further simplified by rearranging orders of factors and orders of integration.

$$\overline{\tilde{V}_s^*(\omega)\tilde{V}_s(\omega)}=\left(\frac{\tau}{2\pi}\right)^2\Bigg(\overline{N}\iiint_\Omega d\mathbf{r}'\int_{-\infty}^{\infty}d\omega'\ T^*(\omega')B_0^*(\omega')\psi^*(\omega')\mathrm{sinc}[(\omega-\omega')(\tau/2\pi)]\,[A_0^*(\mathbf{r}',\omega')]^2$$
$$\times\int_{-\infty}^{\infty}d\omega''\ T(\omega'')B_0(\omega'')\psi(\omega'')\mathrm{sinc}[(\omega-\omega'')(\tau/2\pi)]\,[A_0(\mathbf{r}',\omega'')]^2$$

$$+\overline{N}^2\int_{-\infty}^{\infty} d\omega'\, T^*(\omega')B_0^*(\omega')\psi^*(\omega')\text{sinc}[(\omega-\omega')(\tau/2\pi)]\iiint_\Omega d\mathbf{r}'[A_0^*(\mathbf{r}',\omega')]^2$$

$$\left.\times\int_{-\infty}^{\infty} d\omega''\, T(\omega'')B_0(\omega'')\psi(\omega'')\text{sinc}[(\omega-\omega'')(\tau/2\pi)]\iiint_\Omega d\mathbf{r}''[A_0(\mathbf{r}'',\omega'')]^2\right)$$

$$=\left(\frac{\tau}{2\pi}\right)^2\left(\overline{N}\iiint_\Omega d\mathbf{r}'\,\|J_\omega(\mathbf{r}')\|^2+\overline{N}^2\|M_\omega\|^2\right), \tag{12}$$

where

$$J_\omega(\mathbf{r}')\equiv\int_{-\infty}^{\infty} d\omega'\, T(\omega')B_0(\omega')\psi(\omega')\text{sinc}[(\omega-\omega')(\tau/2\pi)]\,[A_0(\mathbf{r}',\omega')]^2 \tag{13}$$

and

$$M_\omega\equiv\int_{-\infty}^{\infty} d\omega'\, T(\omega')B_0(\omega')\psi(\omega')\text{sinc}[(\omega-\omega')(\tau/2\pi)]\iiint_\Omega d\mathbf{r}'[A_0(\mathbf{r}',\omega')]^2. \tag{14}$$

To separate out the differential scattering cross section per particle at 180°, $\|\psi(\omega')\|^2$, we are faced with the problem that, in $J_\omega(\mathbf{r}')$ and M_ω, $\psi(\omega')$ appears in the integrand where ω' is a dummy variable of integration. If the pulse and gate durations are long enough, however, there should be a very strong peaking of $J_\omega(\mathbf{r}')$ and M_ω at ω_0, the carrier frequency of the pulse; thus, one method of dealing with the problem is to assure that we use narrow banded pulses and sufficiently large gate durations. In general, the frequency dependence of $\psi(\omega')$ will then be small enough that, for $\omega=\omega_0$, $\psi(\omega')\approx\psi(\omega_0)$, and we have

$$\|J_{\omega_0}(\mathbf{r}')\|^2\approx\|\psi(\omega_0)\|^2\|J'_{\omega_0}(\mathbf{r}')\|^2 \tag{15}$$

and

$$\|M_{\omega_0}\|^2\approx\|\psi(\omega_0)\|^2\|M'_{\omega_0}\|^2, \tag{16}$$

where $J'_{\omega_0}(\mathbf{r}')$ and M'_{ω_0} are the same as $J_{\omega_0}(\mathbf{r}')$ and M_{ω_0}, respectively, with the $\psi(\omega')$ factors missing from the integrands.

The backscatter coefficient, or differential scattering cross section per unit volume at frequency ω_0 and 180° scattering angle, is given by $\eta(\omega_0)=\overline{N}\|\psi(\omega_0)\|^2$. Substituting the approximations (15) and (16) into Eq. (12), we get

$$\eta(\omega_0)\approx\overline{\tilde{V}_s^*(\omega_0)\tilde{V}_s(\omega_0)}\times\left[\left(\frac{\tau}{2\pi}\right)^2\left(\iiint_\Omega d\mathbf{r}'\,\|J'_{\omega_0}(\mathbf{r}')\|^2+\overline{N}\|M'_{\omega_0}\|^2\right)\right]^{-1}. \tag{17}$$

Expression (17) is suitable for determining $\eta(\omega_0)$ only if $\overline{N}$ is known or if the term in the denominator containing $\overline{N}$ is negligible. A requirement that $\overline{N}$ be known is inconvenient, particularly for making measurements on tissues. As noted earlier, however, for sufficiently long gate durations τ,

$$\overline{N}\|M'_{\omega_0}\|^2\Big/\iiint_\Omega d\mathbf{r}'\|J'_{\omega_0}(r')\|^2\underset{\tau\text{ increasing}}{\longrightarrow}0,$$

or

$$\eta(\omega_0)\underset{\tau\text{ increasing}}{\longrightarrow}\overline{\tilde{V}_s^*(\omega_0)\tilde{V}_s(\omega_0)}\times\left[\left(\frac{\tau}{2\pi}\right)^2\iiint_\Omega d\mathbf{r}'\,\|J'_{\omega_0}(\mathbf{r}')\|^2\right]^{-1}.$$

Thus, if one plots the ratio of $\overline{\tilde{V}_s^*(\omega_0)\tilde{V}_s(\omega_0)}$ (measured) to

$$\left(\frac{\tau}{2\pi}\right)^2\iiint_\Omega d\mathbf{r}'\,\|J'_{\omega_0}(\mathbf{r}')\|^2$$

vs τ, this ratio should decrease to the asymptotic constant value, $\eta(\omega_0)$, making knowledge of $\overline{N}$ unnecessary.

Once $\eta(\omega_0)$ has been determined in this fashion, an estimate of $\overline{N}$ can itself be obtained by solving (17) for $\overline{N}$:

$$\overline{N}=\|M'_{\omega_0}\|^{-2}\left((2\pi/\tau)^2\eta^{-1}(\omega_0)\,\overline{\tilde{V}_s^*(\omega_0)\tilde{V}_s(\omega_0)}-\iiint_\Omega d\mathbf{r}'\,\|J'_{\omega_0}(\mathbf{r}')\|^2\right). \tag{18}$$

Equation (18) is useful if the expectation value is obtained for sufficiently *small* values of τ that the coherent effects are significant.

Recall that $\|M'_{\omega_0}\|$ and

$$\iiint_\Omega d\mathbf{r}'\,\|J'_{\omega_0}(r')\|^2$$

are functions of τ via the sinc function in the integrands and that the experimental values of $\overline{\tilde{V}_s^*(\omega_0)\tilde{V}_s(\omega_0)}$ depend on (increase with) τ. Ideally, each value of τ used would give the same value for $\overline{N}$. $\overline{N}$ for a complex system, such as tissue, is not known. The above procedure for determining $\eta(\omega_0)$ *and* $\overline{N}$, however, may prove useful in generating a new ultrasonic tissue parameter, viz., $\overline{N}$. In the case of liver, it has been proposed that there are two sets of scatterers,[8] one set being in the size or spacing range of a millimeter, and the other set being in the range of 20–40 μ. Because of the fourth power dependence on frequency of Rayleigh scattering, the backscatter coefficient at low frequencies (e.g., 1 MHz) would depend on the concentration of larger scatterers and that at higher frequencies would depend on the concentration of the small (Rayleigh) scatterers. Thus, $\overline{N}$ would reasonably have two values depending on the frequency.

Another interesting application of the above discussion is that $\eta(\omega_0)/\overline{N}$ would yield a differential scattering cross section per scatterer at 180° and at angular frequency ω_0.

Whether tissues are simple enough in structure to be representable as a two set system of scatterers, each set being composed of identical scatterers, is questionable. In any event, accurate or not, the generated values of $\overline{N}$ and $\eta(\omega_0)/\overline{N}$ for low frequencies and high might be useful diagnostic parameters.

C. Application to broadband pulses

Some measurements of backscatter coefficients, $\eta(\omega)$, reported in the literature, involve use of broadbanded pulses.[5,6,8] In these cases it is assumed that in the data reduction, one needs to be concerned only with the one specific frequency component in any one analysis. Section II A makes it clear that, for pulses with very broad frequency bands, such an assumption deserves scrutiny and that, to get a value of $\eta(\omega)$ which is truly independent of other frequencies, one needs to properly account for the influence of other frequencies on the data. Particular attention needs to be paid to points on spectral distribution curves where the magnitude of the slope is large.

The method of data reduction in Sec. II A may find its greatest use in evaluating $\eta(\omega)$ for a range of frequencies when broadbanded pulses are employed. Very importantly, when analyzed properly, data acquired using broadband pulses can yield values of $\eta(\omega)$ over a range of frequencies whereas, for the same data acquisition time, the backscatter coefficient at only one frequency could reasonably be obtained using a narrow-band pulse.

For data analysis in which broadband pulses are employed, use of an iterative method is proposed. Such a method is described below.

Equation (12) will be used as a starting point in developing the iterative method for determining $\eta(\omega)$, viz.,

$$\overline{\tilde{V}_s^*(\omega)\tilde{V}_s(\omega)} = \left(\frac{\tau}{2\pi}\right)^2\left(\overline{N}\int\int_\Omega\int d\mathbf{r}'\,\|J_\omega(\mathbf{r}')\|^2 + \overline{N}^2\|M_\omega\|^2\right), \quad (12')$$

where $J_\omega(\mathbf{r}')$ and M_ω are given in (13) and (14). $\psi(\omega')$ appears in the integrands of $J_\omega(\mathbf{r}')$ and M_ω, where ω' is a dummy variable. It is proposed that a guess be made at the frequency dependence of $\psi(\omega')$ and that this frequency dependence be separated out as a factor, $g(\omega')$: $\psi(\omega')\equiv\psi_0(\omega')g(\omega')$. The quantity $\psi_0(\omega')$ will be a more slowly varying function of ω' than $\psi(\omega')$ if $g(\omega')$ was judiciously chosen; e.g., for scatterers which are thought to be small compared to the wavelength (Rayleigh-like), one might propose $g(\omega')=\omega'^2$. [For true Rayleigh scatterers, $\psi_0(\omega')$ is then independent of ω', of course.] In our development we ignore any frequency-dependent phase factor of ψ and pursue only the frequency dependence of $\|\psi\|$.

Now suppose that τ is made large enough that the coherent ($\overline{N}^2$) term is negligible in Eq. (12). (This can be determined experimentally in the fashion described in Sec. II A.) Also, we take $\psi_0(\omega')$ to be nearly constant in the neighborhood of ω and move it outside the integral sign in $J_\omega(\mathbf{r}')$; i.e.,

$$J_\omega(\mathbf{r}')\approx J_{\omega,L}(\mathbf{r}')\equiv\psi_0(\omega)K_{\omega,L}(\mathbf{r}'),$$

where

$$K_{\omega,L}(\mathbf{r}')\equiv\int_{-\infty}^{\infty} d\omega'\,T(\omega')B_0(\omega')g(\omega') \times \mathrm{sinc}\,[(\omega-\omega')(\tau_L/2\pi)]\,[A_0(\mathbf{r}',\omega')]^2.$$

The subscript L refers to a long gate duration. ψ_0 is designated above as a function of ω due to the facts that there will in general be residual frequency dependence in ψ_0 and that there is some peaking in the integrand of $J_\omega(\mathbf{r}')$ at ω due to the sinc function. Thus, Eq. (12) becomes

$$\overline{\tilde{V}_s^*(\omega)\tilde{V}_s(\omega)}|_L \approx\left(\frac{\tau_L}{2\pi}\right)^2\overline{N}\,\|\psi_0(\omega)\|^2\int\int_\Omega\int d\mathbf{r}'\,\|K_{\omega,L}(\mathbf{r}')\|^2$$

or

$$\overline{N}\,\|\psi_0(\omega)\|^2\approx\overline{\tilde{V}_s^*(\omega)\tilde{V}_s(\omega)}|_L \times\left[\left(\frac{\tau_L}{2\pi}\right)^2\int\int_\Omega\int d\mathbf{r}'\,\|K_{\omega,L}(\mathbf{r}')\|^2\right]^{-1}.$$

A plot of $(\overline{N})^{1/2}\|\Psi_0(\omega)\|$ vs ω (a set of ω's must be chosen from the frequency band, of course) then yields, via curve fitting, a frequency dependence of $\|\psi_0(\omega)\|$; i.e., we can determine a real function f such that $\psi_0(\omega')=\psi_{00}f(\omega')$, where ψ_{00} likely is nearly independent of frequency and f is a real function of frequency. This completes the first iteration with $\psi(\omega)\,\tilde{\propto}\,f(\omega)g(\omega)$.

The second iteration is accomplished by repeating the above process, yielding

$$\overline{N}\,\|\psi_{00}(\omega)\|^2\approx\overline{\tilde{V}_s^*(\omega)\tilde{V}_s(\omega)}|_L \times\left[\left(\frac{\tau_L}{2\pi}\right)^2\int\int_\Omega\int d\mathbf{r}'\,\|K'_{\omega,L}(\mathbf{r}')\|^2\right]^{-1},$$

where

$$K'_{\omega,L}(\mathbf{r}')\equiv\int_{-\infty}^{\infty} d\omega'\,T(\omega')B_0(\omega')f(\omega')g(\omega') \times\mathrm{sinc}\,[(\omega-\omega')(\tau_L/2\pi)]\,[A_0(\mathbf{r}',\omega')]^2. \quad (19)$$

It may result (and seems likely) that $\psi_{00}(\omega)$ will be almost frequency independent. If so, we can use the results of the first iteration to express $\eta(\omega)$, viz.,

$$\eta(\omega)\approx\overline{N}\,\|\psi_{00}\|^2f^2(\omega)g^2(\omega) = \overline{\tilde{V}_s^*(\omega)\tilde{V}_s(\omega)}|_L\,f^2(\omega)g^2(\omega)$$

$$\times\left[\left(\frac{\tau_L}{2\pi}\right)^2 \iiint_\Omega d\mathbf{r}' \|K'_{\omega,L}(\mathbf{r}')\|^2\right]^{-1}. \quad (20)$$

Otherwise, another iteration could be performed.

The mean particle concentration $\overline{N}$ can also be determined in a way analogous to that described in Sec. II A. Let us assume for this discussion that ψ_{00} can be considered constant and, therefore, that Eq. (20) represents a good approximation for $\eta(\omega)$. Then, with this knowledge of $\eta(\omega)$, a determination of $\overline{\tilde{V}_s^*(\omega)\tilde{V}_s(\omega)}$ for a short enough gate duration, τ_S, will cause the $\overline{N}^2$ term to be significant in Eq. (12) and $\overline{N}$ can be determined using the expression

$$\overline{N} = \|N'_{\omega,S}\|^{-2}\left[\left(\frac{2\pi}{\tau_S}\right)^2 \eta^{-1}(\omega)\, \overline{\tilde{V}_s^*(\omega)\tilde{V}_s(\omega)}|_S \times f^2(\omega)g^2(\omega) - \iiint_\Omega d\mathbf{r}' \|K'_{\omega,S}(\mathbf{r}')\|^2\right], \quad (21)$$

which is obtained from Eq. (12) using the relation

$$\eta(\omega) = \overline{N}\,\|\psi_{00}\|^2 f^2(\omega)g^2(\omega)$$

and the definitions

$$K'_{\omega,S}(\mathbf{r}') \equiv \int_{-\infty}^{\infty} d\omega'\, T(\omega')B_0(\omega')f(\omega')g(\omega') \times \mathrm{sinc}\,[(\omega-\omega')(\tau_S/2\pi)]\,[A_0(\mathbf{r}',\omega')]^2$$

and

$$N'_{\omega,S} \equiv \int_{-\infty}^{\infty} d\omega'\, T(\omega')B_0(\omega')f(\omega')g(\omega') \times \mathrm{sinc}\left[(\omega-\omega')\left(\frac{\tau_S}{2\pi}\right)\right] \iiint_\Omega d\mathbf{r}'[A_0(\mathbf{r}',\omega')]^2,$$

where the subscripts "S" refer to a "short" gate duration.

III. A PRELIMINARY TEST OF THE ACCURACY OF THE METHOD

The method of data reduction applies for focused or nonfocused transducers and for gated volumes at essentially any chosen distance from the transducer face. Extensive tests of the method are reported as part of an article in preparation. In this section we present a few preliminary tests of the method by making comparisons of backscatter coefficients measured on a well defined system of scatterers with values calculated from basic principles.

The scatterers consist of glass spheres with a strong peak in their diameter distribution at 41 μ. A histogram of the diameter distribution is shown in Fig. 3. This shows that the diameter distribution is narrow.

These spheres were randomly distributed in a test cylinder containing a molten agar gel congealed during slow rotation about a horizontal axis (to prevent gravitational sedimentation). The mean concentration of scatterers is $\overline{N} = 45.9\times10^3$ scatterers per cm^3. The agar cylinder containing the glass beads has a diameter of 9 cm and a length of 6 cm.

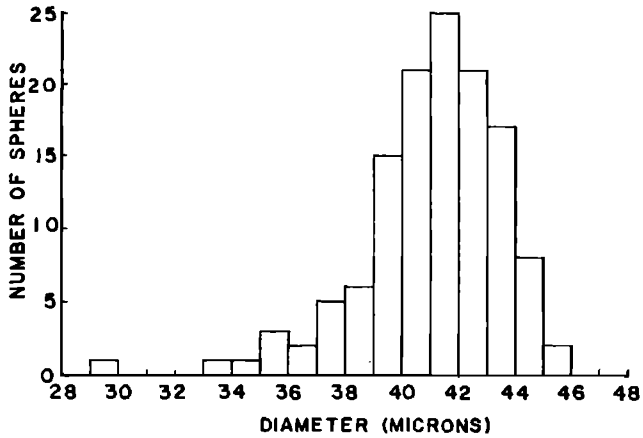

FIG. 3. Histogram showing the diameter distribution of the scatterers in the test sample. The mean diameter is 40.9 μ.

A scattering theory described by Faran,[22] was employed to find the differential scattering cross section per particle at scattering angle of 180°. We write this symbolically as $(d\sigma/d\Omega)|_{180°}$. This theory accounts for both longitudinal and transverse ultrasonic waves and has been found to agree with experiment for direct measurements of differential scattering cross sections over a broad range of scattering angles and frequencies.[3]

Assuming incoherent scattering only, the theoretical backscatter coefficient, or differential scattering cross section per unit volume at 180° scattering angle, is given by $\eta_F(\omega) = \overline{N}(d\sigma/d\Omega)|_{180°}$, where $\overline{N}$ and ω are, of course, the number of particles per unit volume and the angular frequency as in Sec. II; the subscript F denotes "Faran."

Backscatter data were analyzed in two ways. In one a slightly modified version of the Sigelmann and Reid technique[23] was used, the difference being that the reference planar reflector was placed midway between the transducer and the scattering volume instead of at the position of the scattering volume. The other method for data analysis was that described in Sec. II A of this work. In Table I are shown values of $\eta(\omega)$ using the Faran theory, and the values ob-

TABLE I. Backscatter coefficients in cm^{-1} sr^{-1} obtained in various ways for a phantom having well-defined scatterers with a known particle concentration: $\eta_F(\omega)$ is the backscatter coefficient as determined using the Faran[22] theory for single (spherical) scattering; $\eta_{MIZ}(\omega)$ is that determined using the method of data reduction described in Sec. II A of the present paper; $\eta_{MSR}(\omega)$ is that determined using the modified Sigelmann–Reid technique.[23] The uncertainties correspond to standard deviations of the means, also referred to as standard errors.

Frequency (MHz)	$\eta_F(\omega)\times10^5$ (cm^{-1} sr^{-1})	$\eta_{MIZ}(\omega)\times10^5$ (cm^{-1} sr^{-1})	$\eta_{MSR}(\omega)\times10^5$ (cm^{-1} sr^{-1})
1.0	0.328	0.312 ± 0.066	0.194 ± 0.039
1.2	0.678	0.667 ± 0.133	0.281 ± 0.051
2.0	5.10	5.33 ± 1.08	2.32 ± 0.49
2.5	12.2	16.3 ± 2.9	8.5 ± 1.5
3.0	24.5	21.6 ± 3.8	17.2 ± 3.8
4.0	72.4	64.7 ± 13.6	43.7 ± 8.8
5.0	161	155 ± 28	118 ± 26
6.0	300	280 ± 50	240 ± 46

tained by analyzing the data in the two ways described above. Excellent agreement with the Faran result is seen to exist for the method described in this article, and poorer agreement for the case in which the modified Sigelmann–Reid technique was employed.

Experimental parameters were similar for both versions of measurement. The transducer diameter was 12.7 cm, and the distance between the transducer face and the "center" of the gated region was 20 cm. For the modified Sigelmann–Reid technique a pulse duration of 25 μs and gate duration of 10 μs were used. For the version described in this paper, the pulse duration was 10 μs and the gate duration was 25 μs.

The major reason for presenting the data in Table I is to show that the data reduction method described in this paper has passed initial experimental tests.

IV. DISCUSSION AND SUMMARY

A method for reducing data in the form of gated backscatter signals from systems of ultrasonic scatterers which are randomly distributed in space to obtain backscatter coefficients has been derived and tested experimentally. The method makes few approximations, its main feature being that the time gating employed experimentally to select a region of interrogation is dealt with in an exact fashion. Also, the availability of rapidly calculable pressure pulse wave forms facilitates further faithfulness to the actual experimental situation.

The method is applied to the case of randomly positioned scatterers the concentration of which is small enough that coherent backscattering which is not related to the boundaries of the interrogated region is negligible. It is shown that, for sufficiently long gate durations, the backscatter coefficient can be accurately determined without knowledge of the mean concentration of scatterers. Once the backscatter coefficient has been found using a sufficiently long gate duration, shorter gate durations can be employed to determine the mean concentration of scatterers and the differential scattering cross section per particle at 180°. Such a set of measurements may find diagnostic use in patient imaging.

The method of data reduction was tested using a well-defined phantom and found to give excellent agreement with values calculated from basic principles. In a future paper, applications will be presented for various measurement situations. These include use of focused and nonfocused transducers, narrow- and broadbanded frequency spectra, and a variety of gate durations and onset times.

ACKNOWLEDGMENTS

This work was supported in part by NIH grants R01-CA25634 and R01-GM30522.

APPENDIX

In this section we present an argument that, when the (rectangular) gate duration exceeds the pulse duration, the ratio of the coherent term in Eq. (12) to the incoherent term decreases with increasing gate duration, the pulse duration remaining fixed. An extension of the argument is then made that, when the pulse duration exceeds the gate duration, that ratio also decreases as the pulse duration increases, the gate duration remaining fixed; this assumes that the gate is rectangular and that the pulse is of the "continuous wave burst" type.

Consider the first case, viz., that when the gate duration exceeds the pulse duration. Referring to Eq. (12), what we are comparing are the ratios

$$\bar{N}^2\|M_\omega\|^2/\bar{N}\int\int_{\Omega}\int d\mathbf{r}'\,\|J_\omega(\mathbf{r}')\|^2$$

as the gate duration increases. The gate duration will determine the axial extent of the volume of scatterers interrogated. This duration and the volume interrogated will be approximately proportional to one another.

The incoherent term contains an integral over volume of a real positive quantity, viz., $\|J_\omega(\mathbf{r}')\|^2$; thus, this term increases monotonically with the volume of scatterers interrogated. If the lateral pressure amplitude profile does not change much for different distances from the transducer in the region interrogated, then this incoherent term will increase approximately proportionally to the increase in the volume of the region interrogated or, alternatively, to the gate duration.

Now consider the coherent term. In this case phase plays a definite role in that the integral over volume,

$$D\equiv\int\int_{\Omega}\int d\mathbf{r}'[A_0(\mathbf{r}',\omega')]^2,$$

contains a strongly phase-dependent integrand. This factor appears in M_ω defined in Eq. (14). As in the case of the integral over volume in the incoherent term, the part of Ω ultimately contributing is determined by the times of onset and termination of the gate.

Consider an evaluation of D by dividing the volume integrated over into thin volume elements each having a proximal and a distal surface, all points on the proximal surface having the same phase angle θ, and all points on the distal surface having the same phase angle $\theta+\Delta\theta$. These phase angles correspond to those in $[A_0(\mathbf{r}',\omega')]^2$, of course. For ease of visualization, $\Delta\theta$ might be taken to be 1°, for example. For an axially symmetric radiating element, such as in the case of the usual focused or nonfocused transducer in which there are no flaws, the surfaces described will themselves be axially symmetric. Now consider another thin volume element bounded by the proximal surface having phase angle $\theta+180°$ and distal surface having phase angle $\theta+180°+\Delta\theta$. The distance between these two volumetric slabs will be about a quarter of the wavelength corresponding to ω' (not a half-wavelength, because of the backscatter idea involved). For example, if ω' corresponds to 3 MHz, this separation in a tissuelike material would be about 0.125 mm. Thus, the integral of $\|A_0(\mathbf{r}',\omega')\|^2$ over each slab would be nearly the same for both slabs; the phase angles for each slab would be different by 180°, however, and the total contribution to D due to the two slabs would be nearly zero. We can consider the central region of the volume interrogated to be composed primarily of such canceling slabs. Then the major contribution to D would arise at the proximal and distal ends of the interrogated volume where incomplete cancellation could occur, depending on the detailed nature of the onset and termination, respectively, of the gate and on the detailed nature of the pulse wave form.

Thus, the remaining ingredient of our argument is complete, i.e., that the value of the coherent term in Eq. (12) is dependent on the nature of the onset and termination of the gate and not significantly on the gate duration. This idea, combined with the idea that the incoherent term *does* increase with gate duration, means that the relative importance of the coherent term in Eq. (12) in determining the backscatter coefficient decreases with increasing gate duration and for sufficiently long gate duration, is insignificant.

The same argument can be made for the case in which the rectangular gate duration remains fixed and smaller than the pulse duration, and the pulse duration increases (we restrict ourselves to "continuous wave burst" pulses); i.e., for a sufficiently long pulse duration, the coherent term becomes insignificant. In this case, the volume of scatterers interrogated is determined primarily by the pulse duration.

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[15] P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (McGraw-Hill, New York, 1968), Chap. 8, pp. 336, 425. {Note particularly that $j_m(ka) = 1/2[h_m(ka) + h_m^*(ka)]$, where a is the radius of the spherical scatterer.}

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