Scattering by an arrangement of eccentric cylinders embedded on a coated cylinder with applications to tomographic density imaging (L)

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The solution to the scattering of an incident pressure wave by an arrangement of eccentric cylinders embedded inside a pair of concentric cylinders is derived here using a combination of $T$-matrix and mode-matching approaches. This method allows the generation of synthetic data from relatively complex structures to be used for the validation of acoustic tomography methods. An application of the solution derived here is illustrated by reconstructing sound speed and density profiles from a complex phantom using inverse scattering. © 2010 Acoustical Society of America.

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I. INTRODUCTION

The scattering of a cylindrical wave by an arrangement of eccentric cylinders is derived here in order to validate inverse scattering routines with scatterers more complex than a single homogeneous cylinder. Works dealing with the scattering by two concentric fluid cylinders,\textsuperscript{1} two\textsuperscript{2} and multiple\textsuperscript{3} rigid parallel cylinders, two\textsuperscript{4} and multiple\textsuperscript{5} fluid parallel cylinders, two eccentric fluid cylinders,\textsuperscript{6,7} and multiple eccentric cylinders embedded in a circular cylinder\textsuperscript{8} can be found in the literature. The work presented here extends the scope of the work in Ref. 8 by studying the scattering by $N$ circular cylinders embedded inside a coated cylinder, as shown in Fig. 1.

The approach presented here is a combination of the $T$-matrix formulation and mode-matching techniques, and considers changes in compressibility, density, and attenuation (unlike the work in Ref. 8 that only considers changes in refractive index).

II. CALCULATION OF THE SCATTERING COEFFICIENTS

In the following derivation, the background has a wave number $k_0$ and acoustic impedance $Z_0$. For the other cylinders, the complex wave numbers $k_n$ and impedances $Z_n$ are defined as $k_n=|\omega/c_n|+i\alpha_n$, and $Z_n=\rho_n c_n/(1+i\alpha_n c_n/\omega)$, where $c_n$, $\rho_n$, and $\alpha_n$ are the speed of sound, density, and attenuation coefficient in the $n$-th subregion. The radius of the $n$-th cylinder is denoted by $a_n$. The wave number, impedance, and outer radius of the coating are denoted by $k_c$, $Z_c$, and $a_c$, respectively. The acoustic field inside cylinder 1 can be written as

$$p_1(\vec{r}) = \sum_{m=-\infty}^{\infty} A_m j_m(k_1 r_o) e^{i m \theta_o} + \sum_{n=2}^{N} \sum_{m=-\infty}^{\infty} B_{p,n} H_m(k_1 r_{on}) e^{i m \theta_{on}}, \quad (1)$$

where $J_m(\cdot)$ is the $m$-th order Bessel function, $H_m(\cdot)$ is the $m$-th order Hankel function of the first kind, $\vec{r}_n=(r_n, \theta_n)$ are the polar coordinates of the observation point, $\vec{r}_o=(r_o, \theta_o)$ are the polar coordinates of the center of the $n$-th cylinder, and $r_{on}=r_o-r_n$. The second sum in Eq. (1) represents the fields produced by the embedded cylinders. The $T$-matrix approach\textsuperscript{9} can be used to relate the amplitudes of the $A_m$ and $B_{p,n}$ terms. First, a matrix $\hat{T}_s$ is used to express the Bessel waves relative to the center of each one of the $N$ embedded cylinders. The matrix $\hat{T}_s$ is composed of $M$ blocks $\hat{T}_s^m$ of size $N \times M_p$, where $M$ and $M_p$ are the number of terms used to expand the first and second infinite sums in Eq. (1). This can be expressed in matrix form as

$$\vec{\hat{e}}_t = \hat{T}_s \cdot \vec{a}, \quad (2)$$

$$\hat{T}_s = \begin{bmatrix} (\hat{T}_s^m)^{-1} \left[ (\hat{T}_s^{m=0})^T (\hat{T}_s^{m=1})^T \cdots \right]^T \end{bmatrix}^T, \quad (3)$$

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\[
\mathbf{T}_{\mathrm{m}}^m = J_{p-m}(k r_n) e^{-(p-m)(\theta_{\phi} + \pi)},
\]
where \( \mathbf{a} \) is an \( M_n \) length vector with the amplitudes of the Bessel wave harmonics and \( \mathbf{e} \) is an \((M \times N)\) vector with the translated amplitudes of the Bessel waves. In the notation above, \( \mathbf{X}^T \) represents the transpose of a matrix \( \mathbf{X} \). The harmonics \( \mathbf{e} \) can be related to the Hankel harmonics in Eq. (1) as
\[
\mathbf{b}_l = [\mathbf{I} - \mathbf{D}(\mathbf{R}) \cdot \mathbf{A}]^{-1} \cdot \mathbf{D}(\mathbf{R}) \cdot \mathbf{e}_l,
\]
where \( \mathbf{b}_l \) is an \((M \times N)\) vector with the \( B_{p,n} \) coefficients and \( \mathbf{D}(\cdot) \) is an operator that transforms a vector into a diagonal matrix. The \((M \times N)\) vector \( \mathbf{R} \) contains the single-cylinder scattering coefficients given by
\[
\mathbf{R} = [(\mathbf{R}^{m=2})^T (\mathbf{R}^{m=3})^T \ldots (\mathbf{R}^{m=N+1})^T]^T,
\]
\[
[\mathbf{R}]_p = \frac{1}{Z_m} J_p(k a_n) J'_p(k a_n) - J'_p(k a_n) J_p(k a_n)
\]
\[
J_p(k a_n) H'_m(k a_n) - \frac{1}{Z_m} J'_p(k a_n) H_m(k a_n)
\]
where the “*” symbol denotes derivative with respect to the total argument and \( Z_m Z_n / Z_1 \). The elements of the \((M \times N) \times (M \times N)\) \( T \)-matrix \( \mathbf{A} \) are given by
\[
\mathbf{A} = \begin{bmatrix}
\bar{A}_{11} & \bar{A}_{12} & \cdots & \bar{A}_{1M} \\
\bar{A}_{21} & \bar{A}_{22} & \cdots & \bar{A}_{2M} \\
\vdots & \vdots & \ddots & \vdots \\
\bar{A}_{M1} & \bar{A}_{M2} & \cdots & \bar{A}_{MM}
\end{bmatrix},
\]
\[
[\bar{\mathbf{A}}_{mn}]_{pq} = \begin{cases} 0 & \text{if } p = q \\ H_{m-n}(k r_p) e^{-(m-n)\theta_{pq}} & \text{else}, \end{cases}
\]
where \((r_{pq}, \theta_{pq})\) is the polar representation of the vector \((r_p - r_q)\) containing the location of the center of cylinder \( p \) relative to the center of cylinder \( q \), with \( p, q \in [2, N+1] \). Enforcing mode-matching at the cylindrical coating becomes simpler if Eq. (1) is expressed in terms of Bessel and Hankel fields centered at the origin. In particular, if the observation point \( r_o \) satisfies \( |r_o| > |r_n|, \forall n \in [2, N+1] \), the addition theorem of Hankel functions allows the pressure field to be expressed as
\[
p_{1}(r_o) = \sum_{m=\infty}^{\infty} (A_m J_m(k r_o) + B_m H^1_m(k r_o)) e^{im\theta_o},
\]
where the relationship between the \( B_m \) and \( B_{p,n} \) coefficients is given by
\[
\mathbf{b} = \mathbf{P} \cdot \mathbf{a},
\]
\[
\mathbf{P} = \mathbf{D}(\mathbf{R}) \cdot \mathbf{T}.
\]
The expression in Eq. (14) is valid, in particular, for points close to the edge of cylinder 1. In the cylindrical coating, the field can be expressed as
\[
p_{1}(r_o) = \sum_{m=\infty}^{\infty} (C_m J_m(k r_o) + D_m H_m(k r_o)) e^{im\theta_o},
\]
where the first sum is the incident field and the second one is the scattered field. The magnitudes of the \( A_m \), \( B_m \), \( C_m \), and \( D_m \) coefficients can be related using the continuity of both pressure and normal particle velocity at \( r_o = a_1 \). Therefore, one can express
\[
\mathbf{D}(\mathbf{R}_{ij}) \cdot \mathbf{a} + \mathbf{D}(\mathbf{R}_{hh}) \cdot \mathbf{b} = \mathbf{c},
\]
\[
\mathbf{D}(\mathbf{R}_{jh}) \cdot \mathbf{a} + \mathbf{D}(\mathbf{R}_{hh}) \cdot \mathbf{b} = \mathbf{d},
\]
where \( \mathbf{c} \) and \( \mathbf{d} \) are \( M_p \) length vectors with the coefficients \( C_m \) and \( D_m \), respectively, and \( \mathbf{R}_{ij}, \mathbf{R}_{hh}, \mathbf{R}_{jh} \), and \( \mathbf{R}_{hh} \) are \( M_p \) length vectors containing the transmission coefficients given by
\[
[\bar{\mathbf{R}}_{jj}]_m = \frac{J_m(k a_1) H'_m(k a_1) - \frac{1}{Z_{rc}} J'_m(k a_1) H_m(k a_1)}{J_m(k a_1) H'_m(k a_1) - J'_m(k a_1) H_m(k a_1)},
\]
\[
[\bar{\mathbf{R}}_{jj}]_m = \frac{H_m(k a_1) H'_m(k a_1) - \frac{1}{Z_{rc}} H'_m(k a_1) H_m(k a_1)}{J_m(k a_1) H'_m(k a_1) - J'_m(k a_1) H_m(k a_1)},
\]
\[
[\bar{\mathbf{R}}_{hh}]_m = \frac{J_m(k a_1) J'_m(k a_1) - \frac{1}{Z_{rc}} J'_m(k a_1) J_m(k a_1)}{J_m(k a_1) H'_m(k a_1) - J'_m(k a_1) H_m(k a_1)},
\]
\[
[\bar{\mathbf{R}}_{hh}]_m = \frac{H_m(k a_1) J'_m(k a_1) - \frac{1}{Z_{rc}} H'_m(k a_1) J_m(k a_1)}{J_m(k a_1) H'_m(k a_1) - J'_m(k a_1) H_m(k a_1)}.
\]
where $Z_{rc}$ is the normalized resistance of the phantom. Similarly, the field in the background can be written as

$$p(r_o) = \sum_{m=-\infty}^{\infty} \left( E_m J_m(k_o r_o) + F_m H_m(k_o r_o) \right) e^{i m \theta_o},$$

where the first term represents the incident field and the second term represents the scattered field. The coefficients $E_m$ are known and depend on the type of illumination used. For example, when using a line source located at $R_i = (x_i, y_i, z_i)$ the coefficients $E_m$ are given by $E_m = H_m^{(1)}(k r_i) e^{-i m \theta_i}$.

The third term in Eq. (18) can be found using Eqs. (19)–(21) replacing $a_i$ by $a_o$, $k_i$ by $k_o$, $k_j$ by $k_i$, and $Z_{rc}$ by $Z_i / Z_o$. Therefore, the $A_m$ coefficients can be found by using

$$\bar{a} = [D(\bar{R}_{ij2}) \cdot \bar{M}_1 + D(\bar{R}_{ij3}) \cdot \bar{M}_2] \cdot \bar{e},$$

$$\bar{M}_1 = D(\bar{R}_{ij}) + D(\bar{R}_{ij}) \cdot \bar{P},$$

$$\bar{M}_2 = D(\bar{R}_{ij}) + D(\bar{R}_{ij}) \cdot \bar{P}.$$ 

Finally, the scattering coefficients $F_m$ can be obtained by using

$$\bar{f} = [D(\bar{R}_{ij2}) \cdot \bar{M}_1 + D(\bar{R}_{ij3}) \cdot \bar{M}_2] \cdot \bar{a}.$$ 

These coefficients $F_m$ are the ones needed to calculate the scattered field using the second term in Eq. (22).

### III. NUMERICAL VALIDATION

An example of the scattering solution derived in this work is shown in Fig. 2. The dimensions, speed of sound, and density contrasts $\Delta c$ and $\Delta \rho$, and $\alpha/k$ ratios of all cylinders are given in Table I. The incident field was produced by a line source located at $x = 300\lambda$. The values of $M$ and $M_o$ for the scattering solution were set to 51 and 121, respectively. The scattered field was also calculated using the numerical solver presented in Ref. 11 with a grid size of $\lambda/20$. The root mean square error between both calculated scattered fields was only 0.4%, which suggests a proper convergence of the solution derived in this manuscript even in the presence of attenuation. The results are shown in Fig. 2.

Although in principle there should be no restrictions on the sizes and acoustic properties of the embedded cylinders as long as they remain nonoverlapping, special care must be taken when reconstructing scatterers consisting of regions

### TABLE I. Properties of the scatterer used for numerical validation of the scattering solution from Sec. II.

<table>
<thead>
<tr>
<th>Cylinder No.</th>
<th>Center position</th>
<th>Radius</th>
<th>$\Delta c$ (%)</th>
<th>$\Delta \rho$ (%)</th>
<th>$\alpha/k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[0.0]</td>
<td>8.0 $\lambda_o$ (outer)</td>
<td>$7.0 \lambda_o$ (inner)</td>
<td>6.0</td>
<td>-6.0</td>
</tr>
<tr>
<td>2</td>
<td>[0.0]</td>
<td>7.0 $\lambda_o$</td>
<td>-2.0</td>
<td>-1.0</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
<td>[0.0, 2.8$\lambda_o$]</td>
<td>1.6 $\lambda_o$</td>
<td>-3.0</td>
<td>5.0</td>
<td>0.15</td>
</tr>
<tr>
<td>4</td>
<td>[-2.4$\lambda_o$, -2.4$\lambda_o$]</td>
<td>2.4 $\lambda_o$</td>
<td>4.0</td>
<td>2.0</td>
<td>0.2</td>
</tr>
<tr>
<td>5</td>
<td>[2.4$\lambda_o$, -2.4$\lambda_o$]</td>
<td>$\lambda_o$</td>
<td>-2.0</td>
<td>-4.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

TABLE II. Mean speed of sound and density values corresponding to the reconstruction of a complex scatterer using the multiple frequency $T$-matrix approach with different $f_{\text{rms}}$ values.

<table>
<thead>
<tr>
<th>Cylinder No.</th>
<th>Center position</th>
<th>Radius</th>
<th>$(\Delta c, \Delta \rho)$</th>
<th>$f_{\text{rms}}$ = $f_o$</th>
<th>$(\Delta c, \Delta \rho)$</th>
<th>$f_{\text{rms}}$ = $f_o / 16$</th>
<th>$(\Delta c, \Delta \rho)$</th>
<th>$f_{\text{rms}}$ = $f_o / 64$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[0.0]</td>
<td>8.0 $\lambda_o$ (outer)</td>
<td>$7.0 \lambda_o$ (inner)</td>
<td>(1.8%, -1.5%)</td>
<td>(1.68%, -2.5%)</td>
<td>(1.68%, -1.31%)</td>
<td>(1.68%, -1.51%)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>[0.0]</td>
<td>7.0 $\lambda_o$</td>
<td>(1.8%, -1.5%)</td>
<td>(-1.78%, 2.88%)</td>
<td>(-1.76%, 2.09%)</td>
<td>(-1.76%, 1.67%)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>[0.0, 2.8$\lambda_o$]</td>
<td>2.0 $\lambda_o$</td>
<td>(2.5%, -2.5%)</td>
<td>(2.36%, -3.56%)</td>
<td>(2.32%, -1.61%)</td>
<td>(2.32%, -2.13%)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>[-2.4$\lambda_o$, -2.4$\lambda_o$]</td>
<td>1.6 $\lambda_o$</td>
<td>(2.5%, -2.5%)</td>
<td>(-2.49%, 4.11%)</td>
<td>(-2.48%, 2.99%)</td>
<td>(-2.49%, 2.34%)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>


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that exhibit markedly different sizes or acoustic contrasts. In these cases, the finite precision of currently available floating-point arithmetic systems may prevent the proper computation and inversion of all the matrices involved in the solution presented in Sec. II.

IV. APPLICATION TO INVERSE SCATTERING PROBLEMS

The scattering solution presented in Sec. II was used to generate synthetic scattered data from a complex object in order to perform density imaging using the multiple frequency $T$-matrix approach. The performance of this method when imaging homogeneous cylindrical objects has been reported previously. Data were generated at frequencies $f_0,f_0/2,f_0/4,\ldots,f_{\min}$ and processed sequentially starting from the minimum frequency $f_{\min}$. Details of the implementation of the algorithm are given in Ref. 12. The properties and mean reconstructed values for all cylinders using $f_{\min}$ values of $f_0$, $f_0/16$, and $f_0/64$ are given in Table II. The reconstructions are shown in Fig. 3.

The speed of sound reconstructions exhibited high numerical accuracy independently of the value of $f_{\min}$. As for the density reconstructions, the maximum reduction in the bias between the ideal and reconstructed density values for a homogeneous circular cylinder of radius $a$ should occur when $k_{\min}a=1$ according to the results in Ref. 12. Therefore, the cylindrical inclusions of radius $1.6\lambda_0$ and $2\lambda_0$ should already exhibit the minimum achievable bias when using $f_{\min}=f_0/16$. However, the bias was reduced even further when using $f_{\min}=f_0/64$. Therefore, these results suggest that the absolute density values of a complex imaging target may not be obtained unless convergence is guaranteed for the overall structure when using the $T$-matrix approach.

V. CONCLUSIONS

The solution for the scattering produced by multiple parallel cylinders embedded inside a coated cylinder taking into account changes in compressibility, density, and acoustic attenuation has been presented. The applicability of the scattering solution presented here was demonstrated by analyzing the convergence of a method for variable density inverse scattering when imaging objects with multiple levels of spatial variations.