Abstract—Ultrasonic tomography using inverse scattering, i.e.,
the distorted Born iterative method (DBIM), allows for the
quantitative image reconstruction of mechanical properties of
materials. The use of multi-frequency information has been
proposed to avoid convergence issues for targets with moderate
speed of sound contrasts $\Delta c$ (i.e., targets for which the excess
phase $\Delta \phi$ when propagating through the object is larger than
$\pi$) but not validated experimentally. Furthermore, DBIM has
to be regularized due to its ill-conditioning. To experimentally
validate DBIM for use in ultrasonic tomography, a systematic
procedure to choose the regularization parameter based on the
Rayleigh quotient iteration was developed and images of objects
with moderate $\Delta c$ were reconstructed. The performance of DBIM
using the developed regularization scheme was studied through:
1) Simulations of a two dimensional (2D) phantom ($\Delta \phi = 2.14\pi$)
with inclusions smaller than a wavelength, 2) Experiments
with a balloon phantom with high $\Delta c$ with scattered data
collected at 0.64 MHz ($\Delta \phi = 0.84\pi$) and 1.2 MHz ($\Delta \phi = 1.67\pi$),
3) Three dimensional (3D) reconstruction of a moderate
contrast sphere ($\Delta \phi = 1.32\pi$). The 2D and 3D multi-frequency
DBIM simulations were successfully stabilized by the proposed
regularization scheme as evidenced by the low reconstruction
mean square errors (MSEs) ($\Delta c$ MSE $= 9\%$ for 2D and 15% for
3D) and the detection of the sub-wavelength inclusions in the
2D reconstruction. In experiments, the measured scattered fields
agreed well with the predicted scattered fields from the phantom
model (MSE $= 4\%$ for 0.64 MHz and MSE $= 7.8\%$ for 1.2 MHz).
The MSE of the reconstructed image using only the experimental
data at 0.64 MHz was 19%. Using the experimental data at 1.2
MHz to refine the 0.64 MHz reconstruction allowed the MSE
to be reduced to 12%, and improved the spatial resolution as
evidenced by the reduced edge blurring.

I. MOTIVATION

Ultrasonic tomography using inverse scattering methods,
i.e., the distorted Born iterative method (DBIM), allows for
quantitative image reconstruction of mechanical properties of
materials. The use of multi-frequency information has been
proposed to avoid convergence issues for targets with moderate
speed of sound contrasts $\Delta c$ (i.e., targets for which the excess
phase $\Delta \phi$ when propagating through the object is larger than
$\pi$) but not validated experimentally. Furthermore, DBIM has
to be regularized due to its ill-conditioning. The present work
introduces a robust, computationally efficient regularization
scheme to be used with DBIM and both simulations and
experimental validation to assess its performance.

II. INVERSE SCATTERING AND THE DBIM

The details of DBIM [1] are presented here for completeness.
For the special case of constant density, the integral wave
equation can be written as

$$p(\vec{r}) = e_a(\vec{r}) + \int \Omega d\vec{r}' \mathcal{O}(\vec{r}', \vec{r}) p(\vec{r}') G_0(\vec{r}, \vec{r}') \tag{1}$$

where $\Omega$ is the computational domain, $p(\vec{r})$ is the acoustical
pressure, $e_a(\vec{r})$ is the incident field caused by a source located
at $\vec{r}_s$, $s = 0, 1, ..., N_s$, $\mathcal{O}(\vec{r}, \vec{r}) = [k^2(\vec{r}) - k_0^2]$ is known as the
object function with $k(\vec{r})$ the wave number function, $G_0(\vec{r}, \vec{r}')$ is
the wave propagation Green’s function, and $\vec{r}$ is the position
vector.

The object is divided into $N$ subscatterers centered at
positions $\vec{r}_m$, $m = 0, 1, ..., N$. Using delta testing functions,
(1) can be evaluated at all points $\vec{r}_m$ which results in the system
of equations

$$\tilde{p}_s = \left[ \mathcal{I} - \tilde{C} \cdot \mathcal{D}(\mathcal{O}) \right]^{-1} \cdot \tilde{e}_a \tag{2}$$

where $\mathcal{D}(\cdot)$ is an operator that transforms an $N\times1$ vector into
an $N\times N$ diagonal matrix, $\mathcal{O}$, $\tilde{p}_s$, and $\tilde{e}_a$ are $N\times1$ vectors
containing the values of the object function and the total and
incident acoustic pressure at points $\vec{r}_m$ when the source is
placed at $\vec{r}_s$, respectively, and $\tilde{C}$ is an $N\times N$ matrix whose
elements are given by $\left[ \tilde{C} \right]_{mn} = G_m(\vec{r}_n)$, with

$$G_m(\vec{r}) = \int d\vec{r}' G_m(\vec{r}', \vec{r}) \cdot b_m(\vec{r}) \tag{3}$$

Using delta testing functions to evaluate (1) at the receiver
positions $\vec{r}_r$, $r = 1, 2, ..., N_r$, the scattered field can be
calculated as

$$\tilde{p}_s^{sc} = \tilde{D} \cdot \mathcal{D}(\mathcal{O}) \cdot \tilde{p}_s \tag{4}$$

where $\tilde{D}$ is an $N_r\times N\times N$ matrix whose elements are equal to
$\left[ \tilde{D} \right]_{rmn} = G_m(\vec{r}_r)$. In contrast, if the object function is
not known then (1) cannot be solved because the acoustic
field inside the scatterer cannot be calculated. Therefore, the
object function must be obtained using an iterative method.
First, a trial object function $\mathcal{O}(m)$ is chosen for which the
corresponding vectors $\tilde{p}_s^{sc}$ and $\tilde{p}_s$ are calculated. Next, the
Regularization parameter $\sigma$ is given by the regularized optimization problem

$$\Delta O(n) = \arg\min_{\Delta O} \|\Delta \hat{p}^{sc} - \hat{F}_s(n) \cdot \Delta O\|^2 + \alpha \|\Delta O\|^2,$$  \hspace{0.5cm} (5)

where $\Delta \hat{p}^{sc}$ is the difference between the predicted $\hat{p}^{sc}$ and measured $\hat{p}^{sc}$ scattered fields, $\alpha$ is the regularization parameter, and $\hat{F}_s(n)$ is the Frechet derivative matrix, which is composed of $N_s$ stacked matrices $\hat{F}_s$ of the form [2]

$$\hat{F}_s = \hat{D} \cdot \{I - \hat{D}(O) \cdot C\}^{-1} \cdot \hat{D}(\hat{p}_s).$$  \hspace{0.5cm} (6)

### III. PROPOSED REGULARIZATION METHOD

Several methods to determine the optimum $\alpha$ for matrix equations are available in the literature, such as the L-curve [3] or the generalized cross validation (GCV) [4]. However, these methods have a large computational cost and potential detrimental effects on the convergence of DBIM. Some other approaches to find $\alpha$ for inverse scattering can be found in the literature [5]–[8], but they depend on heuristic choices of independent parameters and therefore it is not clear how to use them in general scenarios. Therefore, a systematic and yet efficient approach to select the regularization parameter needs to be developed. The initial regularization parameter $\alpha$ is chosen to be comparable to the square of the first singular value $\sigma_0$ of the Frechet derivative matrix. This choice is sufficient to avoid rapid variations in the initial reconstructed profiles, which have been found to be critical to the convergence of inverse scattering algorithms. As the relative residual error (RRE) decreases, the value of the regularization parameter can be relaxed to allow for contributions from higher frequencies. Because of simplicity, the relaxation process used in this work is based on RRE thresholding. The proposed scheme [9] is shown in Table I.

<table>
<thead>
<tr>
<th>RRE</th>
<th>Regularization parameter $\alpha$</th>
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<tbody>
<tr>
<td>0.5$&lt;\text{RRE}$</td>
<td>$\sigma_0^2/2$</td>
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<tr>
<td>$0.25&lt;\text{RRE}\leq0.5$</td>
<td>$\sigma_0^2/20$</td>
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<tr>
<td>$\text{RRE}\leq0.25$</td>
<td>$\sigma_0^2/200$</td>
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### IV. SIMULATIONS AND EXPERIMENTAL RESULTS

Simulations were performed using the synthetic phantom shown in Fig. 1(a) with $\Delta \phi = 2.14\pi$ at the maximum frequency used. The reconstructed image shown in Fig. 1(b) was obtained using two frequencies to avoid divergence. The synthetic measurements were contaminated with 5% zero-mean Gaussian noise. The multi-frequency DBIM simulations were successfully stabilized by the proposed regularization scheme as evidenced by the low reconstruction mean square error (MSE) ($\Delta$ MSE = 9%) and the detection of the subwavelength inclusions.

Experiments were conducted to validate the ability to perform DBIM inversions when dealing with real data collected using two circular, unfocused transducers. The first transducer was used as source and had a fixed position relative to the imaging target. The second transducer was used as receiver and rotated in a circular arc around the sample. Both transducers had a nominal center frequency of 1 MHz and reported radius of 0.0625 inches.
receiver transducer was mounted on a vertical rod, which was attached to an L-shaped mechanical element with adjustable horizontal length. With this arrangement, rotating the vertical segment of the L-shaped element resulted in the transducer describing a circular arc around the balloon phantom with radius given by the length of the horizontal segment. The receiving angle was changed between -60 and 60 degrees.

Two sets of measurements were collected: one without the sample in the water tank to measure the incident field, and one with the sample in the tank to measure the total acoustic field in the presence of the scatterer. The scattered field is obtained by subtracting the incident field from the total field. The three fields are shown in Fig. 3 as a function of the receiving angle.

The scattered fields at frequencies of interest were obtained by taking the Fourier transform of the measured waveforms for all receiver positions. A frequency of 1.2 MHz was chosen for the DBIM reconstructions. At this frequency, \( \Delta \phi \approx 1.6\pi \) according to the phantom model, and therefore frequency hopping was required. A lower frequency of 0.64 MHz, for which \( \Delta \phi \approx 0.85\pi \), was used for the coarse reconstruction. Figure 4 shows the measured scattered fields as a function of the receiver angle. For comparison, the ideal scattered field was also generated using a method of moments (MoM) solver. The MSEs between the measured and expected scattered fields were 4.1% and 7.8% for 0.64 MHz and 1.2 MHz, respectively.

The phantom reconstructions are shown in Fig. 5. For comparison, the DBIM reconstruction using the scattered data generated using the MoM solver was also calculated. Both reconstructions are in very good agreement. The MSE of the reconstructed image using only the experimental data at 0.64 MHz was 19%. Using the experimental data at 1.2 MHz to refine the 0.64 MHz reconstruction allowed the MSE to be reduced to 12%, and improved the spatial resolution as evidenced by the reduced edge blurring. The mean speed of sound and radius of the phantom were accurately reconstructed.

Three dimensional reconstructions of a sphere of radius \( 4\lambda \) and speed of sound contrast of 9% (\( \Delta \phi = 1.32\pi \)) were also obtained using two methods regularized with the scheme presented in this work. The reconstructions are shown in Fig. 6. In the first approach, simulated rectangular transducers focused on elevation with focal number of 4 were used to produce a series of 2D image slices of the object using 2D DBIM. The resulting 3D image reconstruction was rendered by stacking the serial 2D slices. The reconstruction MSE
was 31.6% and significant diffraction effects occurred when the imaging plane did not pass through the center of the sphere. In the second approach, point-like transducers were simulated to produce a fully 3D DBIM image reconstruction using frequency hopping. The final reconstruction MSE was 15% and no significant diffraction effects at the edges of the sphere were observed.

V. CONCLUSIONS

A novel regularization approach for use in DBIM studies was presented. The approach was successfully tested using two and three dimensional simulations, and experiments with limited angular coverage. All these results indicate that the proposed regularization scheme is capable of producing accurate solutions even with reduced k-space coverage, without causing an excessive blurring of the reconstruction. The experimental results presented here are the first experimental validation of convergence of DBIM reconstructions for large contrast ($\Delta \phi > \pi$) with acoustic data using non-ambiguous imaging targets for which the validity of the reconstructions can be assessed.

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REFERENCES