tical at all periodically equivalent locations along the axis, with the result that adjacent repeated segments of the *total fibril* are of essentially constant length, $b_0$. Consequently, these models are to be distinguished from cases in which the period may be variable.[8] Small-angle diffraction data on collagen have not as yet been obtained at sufficient angular resolution to demonstrate that this sort of fibrillar axial distortion is present, but it is in any case of much less importance than the intra-fibrillar filament distortions which produce the readily observed variations in line length.

[8] See R. Hosemann, Z. Physik 127, 16 (1949).

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# Crystal Systems with Low Loss

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Approximate formulas are derived for computing the effect of mechanical resistance on the electrical characteristics of a piezoelectric crystal system for the case of a general reactive termination and low loss. A comparison of values computed on the basis of these formulas and exact calculations is presented.

## I. INTRODUCTION

We consider the effect of the resistive component of the mechanical terminating impedance on the electrical characteristics of a piezoelectric system. We are concerned with the shift of the resonant and the antiresonant frequencies, the magnitudes of the impedances at resonance and antiresonance, and the quality factor as a function of the magnitude of the resistive component of the mechanical terminating impedance. Some previous work in this direction was restricted to the case of a zero reactive termination.[1] The formulas developed herein are applicable to a low loss system vibrating in either longitudinal or thickness mode in which the reactive component is unrestricted.

## II. ANALYSIS

### Table of Symbols

- $A_c$ Area of radiating face of crystal.
- $C$ $\mathrm{Cos}\gamma_0$.
- $C_0$ Clamped capacity of crystal.
- $d$ Piezoelectric constant.
- $f$ Frequency.
- $f_r$ Resonant frequency.
- $\Delta f$ Difference between resonant and antiresonant frequencies.
- $K$ $V_c/L_c$.
- $l_w$ Width of crystal.
- $L_c$ Length of crystal.
- $M$ $\gamma_0\theta$.
- $Q'$ Quality factor, ratio of the impedance at antiresonance and the impedance at resonance.
- $s$ Elastic constant.
- $S$ $\mathrm{Sin}\gamma_0$.
- $V_c$ Velocity of sound in crystal.
- $x$ Coordinate variable.
- $X_a+jY_a$ $Z_{m2}/Z_0$.
- $Y_a$ See definition of $X_a$.
- $Z_e$ Input electrical impedance into crystal.
- $Z_0$ $A_c\rho_c V_c$.
- $Z_{m2}$ Mechanical impedance of load on crystal.
- $Z_{ar}$ Electrical impedance of crystal system at antiresonance.
- $Z_r$ Electrical impedance of crystal system at resonance.

### Greek Symbols

- $\gamma_c$ $\omega L_c/V_c$.
- $\gamma_0$ Value of $\gamma_c$ at resonance for lossless system.
- $\epsilon$ $\gamma_0-\gamma_c$.
- $\theta$ $\varphi^2/Z_0K_0C_0\gamma_0$.
- $\mu$ $[Y_a^2+1+(dY_a/d\gamma)_{\gamma_0}]\epsilon$.
- $\rho_c$ Density of crystal.
- $\varphi$ $=dl_w/s$ for longitudinal mode.
  $=dA_c/sL_c$ for thickness mode.
  $d$ and $s$ in these formulas are not identical in value for the two modes.
- $\omega$ $2\pi f$.

Consider a crystal vibrating in longitudinal or thickness mode and free at one end, $x=0$, but with a load both resistive and reactive on the opposite end, $x=L_c$

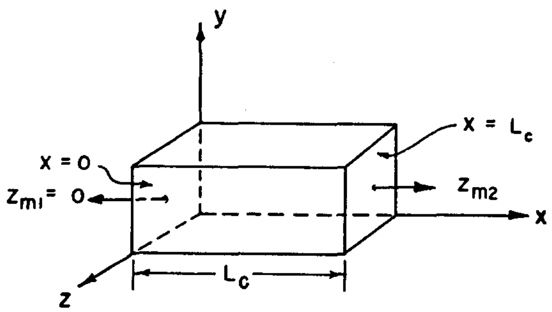


FIG. 1. A crystal placed in a coordinate system.

[1] W. J. Fry, J. Acoust. Soc. Am. 21, 29 (1949).

(see Fig. 1). The general expression for the electrical input impedance under these conditions is[2]

$$Z_e=\frac{1}{j\gamma_c K C_0}+\frac{\varphi^2}{Z_0}\left[\frac{-2(1-\cos\gamma_c)+(jX_a-Y_a)\sin\gamma_c}{(X_a+jY_a)\cos\gamma_c+j\sin\gamma_c}\right]. \quad (1)$$

The resonant and antiresonant frequencies are defined as those frequencies for which the absolute value of the electrical input impedance has its minimum and maximum values.

In (1) insert the relation $\gamma_c=\gamma_0-\epsilon$ and the condition for resonance of a no loss system, $Y_a\cos\gamma_0+\sin\gamma_0=0$, represent $\cos\gamma_c$ by $\cos(\gamma_0-\epsilon)$ and $\sin\gamma_c$ by $\sin(\gamma_0-\epsilon)$; approximate these by $\cos\gamma_0(1-\epsilon^2/2)+\epsilon\sin\gamma_0$ and $\sin\gamma_0(1-\epsilon^2/2)-\epsilon\cos\gamma_0$, respectively; also let

$$Y_a=Y_{a0}-\epsilon(dY_a/d\gamma)_{\gamma_0}+\tfrac{1}{2}\epsilon^2\left(\frac{d^2Y_a}{d\gamma^2}\right)_{\gamma_0}.$$

Form the quantity $|Z_e|^2$. The highest power of $\epsilon$ retained is the second. To obtain the values of $\epsilon$ for which $|Z_e|^2$ has its maximum and minimum, the expression for $|Z_e|^2$ is differentiated with respect to $\epsilon$ and the result equated to zero. Powers up to and including the fifth power of $\epsilon$ are retained in the expression; however, all powers in $X_a$ higher than the lowest in each coefficient are rejected, since the range for $X_a$ is $0\leq X_a\leq 0.05$. Hence, the form

$$\left\{\begin{aligned}&[(1-C)^2C^4\beta^5+\theta(1-C)^2C^2\beta^3E]\epsilon^4\\&+[\theta(1-C)^4C^2\beta^4]\epsilon^3\\&+[3(1-C)^2C^2\beta^3X_a^2SC\theta\\&\qquad-3(1-C)^2C^2\beta EX_a^2\theta\\&\qquad\qquad+6(1-C)^3C^3\beta^2Y_aX_a^2\theta]\epsilon^2\\&+[(1-C)^4C^2\beta^2X_a^2\theta]\epsilon\\&+[2(1-C)^4X_a^4SC\theta-(1-C)^2C^4\beta X_a^4\\&\quad-(1-C)^2C^3\beta X_a^4S\theta-2(1-C)^3C^3Y_aX_a^4\theta]\end{aligned}\right\}=0, \quad (2)$$

where

$$\beta=Y_a^2+1+(dY_a/d\gamma)_{\gamma_0},$$

and

$$E=(C-1)^2[-2Y_a(dY_a/d\gamma)_{\gamma_0}-\tfrac{1}{2}(d^2Y_a/d\gamma^2)_{\gamma_0}]-Y_a[1+(dY_a/d\gamma)_{\gamma_0}C]^2.$$

Although these coefficients appear unwieldy and irreducible, an approximation of the form

$$\mu^4+\theta\left(\frac{1-C}{C}\right)^2\mu^3+\left(\frac{1-C}{C}\right)^2X_a^2\theta\mu-X_a^4=0, \quad (3)$$

gives a result which is accurate to within ten percent for the quantities of interest when $Y_a$ is on the range $0\leq Y_a\leq 2$.

The real roots of this equation correspond to the resonant and antiresonant frequencies. Explicitly,

$$\mu_r=-\left(\frac{C-1}{C}\right)^2\frac{\theta}{2}+\left(\frac{C-1}{C}\right)^2\frac{\theta}{2}\left[1+X_a^2\Big/\left(\frac{1-C}{C}\right)^4\left(\frac{\theta}{2}\right)^2\right]^{\frac{1}{2}}, \quad (4a)$$

$$\mu_{ar}=-\left(\frac{C-1}{C}\right)^2\frac{\theta}{2}-\left(\frac{C-1}{C}\right)^2\frac{\theta}{2}\left[1+X_a^2\Big/\left(\frac{1-C}{C}\right)^4\left(\frac{\theta}{2}\right)^2\right]^{\frac{1}{2}}. \quad (4b)$$

From these results, it follows that

$$\frac{\Delta f}{f_r}=\frac{\{[(1-\cos\gamma_0)/\cos\gamma_0]^4\theta^2+4X_a^2\}^{\frac{1}{2}}}{[Y_a^2+1+(dY_a/d\gamma)_{\gamma_0}]\tan^{-1}(-Y_a)}. \quad (5)$$

By evaluating the electrical input impedance at the resonant and antiresonant frequencies, a quality factor

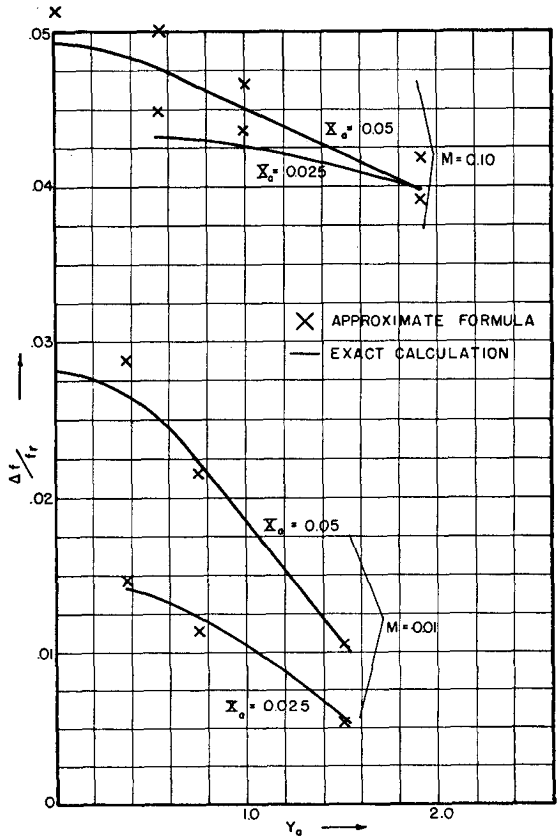


FIG. 2. Comparison of exact calculations with results computed from the approximate formula for the quantity $\Delta f/f_r$.

[2] See Fry, Taylor, and Henvis, *Design of Crystal Vibrating Systems* (Dover Publications, New York, 1948), for one derivation.

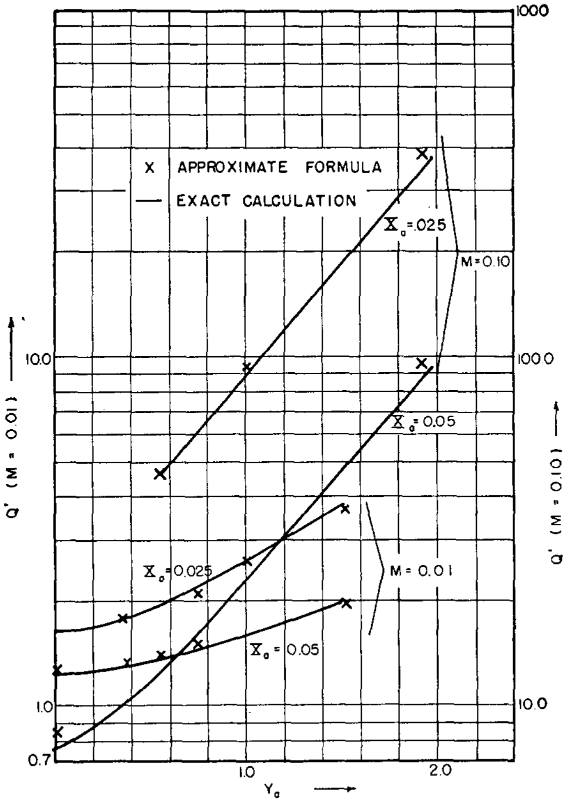


FIG. 3. Comparison of exact calculations with results computed from the approximate formula for the quality factor $Q'$.

is obtained. This quantity is defined as the absolute value of the ratio of the impedance at antiresonance to the impedance at resonance. Using the values found for $\mu_{ar}$ and $\mu_r$ given by expressions (4a) and (4b), and the equation representing $Z_e$, making reductions of form similar to those made to obtain $\mu_r$, $\mu_{ar}$ the representations for $|Z|_r$ and $|Z|_{ar}$ follow:

$$|Z_r|=\frac{([\cos\gamma_0-1]/\cos\gamma_0)^2\frac{1}{2}\theta}{X_aKC_0\gamma_0}\times\left\{-1+\left[1+\frac{X_a^2}{([\cos\gamma_0-1]/\cos\gamma_0)^4(\frac{1}{2}\theta)^2}\right]^{\frac{1}{2}}\right\}, \quad (6a)$$

$$|Z_{ar}|=\frac{([\cos\gamma_0-1]/\cos\gamma_0)^2\frac{1}{2}\theta}{X_aKC_0\gamma_0}\times\left\{1+\left[1+\frac{X_a^2}{([\cos\gamma_0-1]/\cos\gamma_0)^4(\frac{1}{2}\theta)^2}\right]^{\frac{1}{2}}\right\}. \quad (6b)$$

The quality factor, $Q'=|Z_{ar}/Z_r|$, is then obtained directly as

$$Q'=\frac{[1+X_a^2/\{([\cos\gamma_0-1]/\cos\gamma_0)^4(\frac{1}{2}\theta)^2\}]^{\frac{1}{2}}+1}{[1+X_a^2/\{([\cos\gamma_0-1]/\cos\gamma_0)^4(\frac{1}{2}\theta)^2\}]^{\frac{1}{2}}-1}, \quad (7)$$

which is seen to be the ratio of $\mu_{ar}$ to $\mu_r$.

For $Y_a=0$ and $0\leq X_a\leq 0.05$, the quantities $\Delta f/f_r$, $|Z_r|$, $|Z_{ar}|$, and $Q'$ reduce to those given in a previous paper.[1]

For a crystal system, expressions (5) and (7) relate the quality factor, $Q'$, and the quantity $\Delta f/f_r$ to the quantity $X_a$, which is the ratio of the acoustic input resistance into the backing to the characteristic impedance of the crystal. This quantity, $X_a$, characterizes the losses.

## III. COMPARISONS

Because of the large number of steps involved in the derivation in which approximations were introduced, it is somewhat difficult to specify the ranges of applicability of these expressions for a given accuracy in $Q'$ and $\Delta f/f_r$ without some numerical comparisons. Accordingly, a numerical comparison was carried out between the values computed on the basis of these formulas and on the basis of the exact relation (1) from which these expressions were derived. The results of this work appear in Figs. 2 and 3. The specific examples chosen correspond to a quartz mercury system, $M=0.01$, and to an ADP mercury system, $M=0.1$. Calculations were made for four lengths of mercury. The value of $Y_a$ is related to the percentage frequency shift expressed as a percent of the resonant frequency of the free crystal. For the ADP Hg system the maximum shift for which values were computed is 35 percent and for the quartz Hg system 32 percent. Figure 2 indicates the results for $\Delta f/f_r$ and Fig. 3 illustrates the results on the quality factor $Q'$. It should be pointed out that it was not the purpose of this numerical work to obtain accurate numerical estimates of the difference between exact calculations and those using the expressions (5) and (7) for the purpose of adding correction terms, but to obtain only an estimate of the percentage error involved for a range of the variables $X_a$ and $Y_a$. As the result of this analysis, we can state that expressions (5) and (7) can be used for the region of the variables, $Y_a$ and $X_a$, $0\leq Y_a\leq 2$ and $0\leq X_a\leq 0.05$ with an error not exceeding roughly 12 percent in the value $Q'-1$ and an error not exceeding 10 percent in the quantity $\Delta f/f_r$. As $X_a$ approaches zero, the error approaches zero, so that if $X_a$ is further restricted, values calculated on the basis of these formulas are correspondingly more accurate.