Low Loss Crystal Systems*

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Some aspects of the theory of low loss crystal systems are presented. The effect of small losses on the resonant frequency, the difference between the resonant and antiresonance frequencies, and the electrical impedances at resonance and at antiresonance, is calculated. A quality factor is defined and related to the mechanical loads on the crystal.

I. INTRODUCTION

IN this paper we consider the effect of the damping or resistive mechanical loads on the characteristics of a piezoelectric crystal system vibrating in either a longitudinal or thickness mode. In particular, the effect on the resonant and antiresonance frequencies is discussed. Losses in the interior of the crystal are neglected. A quality factor is defined for damped crystal systems. The value of this quality factor is related to the mechanical load impedances by appropriate formulas. The discussion is limited to consideration of systems which are not too heavily damped. The meaning of this statement will become more precise as the theory is developed.

The results are applied to particular systems for the purpose of illustration. Some aspects of the double crystal acoustic interferometer are discussed.

II. GENERAL ANALYSIS

Consider a crystal vibrating either in a longitudinal or a thickness mode, and loaded on each end with mechanical impedances. See Fig. 1, in which the thickness mode is illustrated.

The crystal is placed in a coordinate system with the left face at $x=0$ and the right face at $x=L_c$. If the thickness of a crystal vibrating in a thickness mode is small compared to the other dimensions, or if the length of a crystal excited in a longitudinal mode is large compared to the other dimensions, a one-dimensional theory is applicable.[1]

We proceed with the development of the theory. For the purpose of this discussion the resonant and antiresonant frequencies of a crystal system can be defined as the frequencies at which the absolute value of the electrical input has its minimum and maximum values, respectively.

The general expression for the electrical input impedance is (p. 156 of *Design of Crystal Vibrating Systems*, hereafter abbreviated CVS)[1]

$$Z_e=1/[j\omega C_0+\varphi^2(\beta_1-\beta_2)], \tag{1}$$

where

$$\frac{1}{\beta_1-\beta_2}=Z_0\frac{\{(Z_1/Z_0+Z_2/Z_0)\cos\gamma_c+j\sin\gamma_c[(Z_1/Z_0)(Z_2/Z_0)+1]\}}{-2(1-\cos\gamma_c)+j\sin\gamma_c[(Z_1/Z_0)+(Z_2/Z_0)]}. \tag{2}$$

We consider the case of purely resistive load impedances, that is, Z_1 and Z_2 are real positive quantities. We further assume that the magnitudes of Z_1 and Z_2 are such that the approximations used in the development are fairly good. The limitations on Z_1/Z_0 and Z_2/Z_0 will become clear as the development proceeds.

The quantity Z_0 is real and positive, as can be seen immediately from its definition, $Z_0=A_c\rho_cV_c$. Therefore the expressions $(Z_1/Z_0+Z_2/Z_0)$ and $[(Z_1Z_2/Z_0Z_0)+1]$ are real and positive. It is convenient to introduce the following notation:

$$A=(Z_1/Z_0+Z_2/Z_0), \tag{3}$$

and

$$B=[(Z_1/Z_0)(Z_2/Z_0)+1].$$

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[1] See, for example, W. J. Fry, J. M. Taylor, and B. W. Henvis, *Design of Crystal Vibrating Systems* (Dover Publications, New York, 1948).

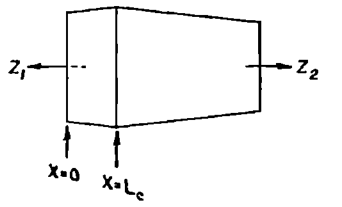

FIG. 1. A cut crystal and associated coordinate and impedance notation.

The expression for $1/\beta_1-\beta_2$ then becomes:

$$1/\beta_1-\beta_2=Z_0(A\cos\gamma_c+jB\sin\gamma_c/ \\ -2(1-\cos\gamma_c)+jA\sin\gamma_c). \quad (4)$$

This paper is concerned with a study of the characteristics of the crystal system for frequencies in the neighborhood of the first resonant frequency of the undamped free crystal. This resonant frequency is given by the expression[2] $\gamma_c=\pi$.

Although this discussion is restricted to frequencies in the neighborhood of the first resonant frequency, a study of the characteristics of a crystal system in a neighborhood of any resonant frequency would proceed in the same fashion.

It is convenient to introduce the quantity ϵ as follows: $\gamma_c=\pi-\epsilon$.

The expression for electrical input impedance will be considered for small values of ϵ.

For the purposes of this discussion the function $\sin\gamma_c$ and $\cos\gamma_c$ can be approximated by the expressions,

$$\begin{aligned}\sin\gamma_c&=\sin\epsilon\simeq\epsilon-\epsilon^3/\underline{|3},\\ \cos\gamma_c&=-\cos\epsilon\simeq-[1-\epsilon^2/\underline{|2}+\epsilon^4/\underline{|4}].\end{aligned} \quad (5)$$

When these relations are substituted into expression (4) the following form for $\beta_1-\beta_2$ results:

$$\beta_1-\beta_2=\frac{1}{Z_0}\frac{-2[2-\epsilon^2/\underline{|2}+\epsilon^4/\underline{|4}]+jA[\epsilon-\epsilon^3/\underline{|3}]}{-A[1-\epsilon^2/\underline{|2}+\epsilon^4/\underline{|4}]+jB[\epsilon-\epsilon^3/\underline{|3}]}. \quad (6)$$

Upon rationalizing (6) and retaining powers of ϵ up to and including the fourth, we obtain:

$$\beta_1-\beta_2=\frac{1}{Z_0}\frac{\{A[4-3\epsilon^2+3\epsilon^4/4]+AB[\epsilon^2-\epsilon^4/3]\}+j\{A^2[-\epsilon+2\epsilon^3/3]+B[4\epsilon-5\epsilon^3/3]\}}{A^2[1-\epsilon^2+\epsilon^4/3]+B^2[\epsilon^2-\epsilon^4/3]}. \quad (7)$$

Let us now restrict the discussion to the case of $A\leq 0.1$. Then the value of B is 1 to within 1 percent. To the degree of accuracy in which we are interested B can then be taken equal to one. If we set $B=1$ in (7) and substitute the result into (1), the expression for Z_e becomes the following after collecting terms:

$$Z_e=\frac{A^2+\epsilon^2(1-A^2)-(\epsilon^4/3)(1-A^2)}{(\varphi^2/Z_0)A[4-2\epsilon^2+5\epsilon^4/12]+j\{\omega C_0[A^2+\epsilon^2(1-A^2)-\epsilon^4(1-A^2)/3]+(\varphi^2/Z_0)[\epsilon(4-A^2)+\epsilon^3(2A^2-5)/3]\}}. \quad (8)$$

The square of the absolute value of Z_e is then

$$|Z_e|^2=\frac{[A^2+\epsilon^2(1-A^2)-\epsilon^4(1-A^2)/3]^2}{(\varphi^2/Z_0)^2A^2[4-2\epsilon^2+5\epsilon^4/12]^2+\{\omega C_0[A^2+\epsilon^2(1-A^2)-\epsilon^4(1-A^2)/3]+(\varphi^2/Z_0)[\epsilon(4-A^2)+\epsilon^3(2A^2-5)/3]\}^2}. \quad (9)$$

To obtain the values of ϵ for which this function is a maximum and a minimum, we can differentiate with respect to ϵ and set the resulting expression equal to zero. The positions of the maxima and minima of $|Z_e|^2$ are identical with those of $|Z_e|$ since $(d/d\epsilon)|Z_e|^2=2|Z_e|(d|Z_e|/d\epsilon)$ and $|Z_e|$ is always greater than zero. We assume that A is independent of ϵ over the range of values of interest. The quantity ω is a function of ϵ given by $\omega=(\pi-\epsilon)V/L$. However, it can be shown that for the ranges of the variables of interest the terms arising from differentiation of ω cause a change of only a few percent in the end

[2] See, for example, reference 1, p. 163.

results. The inclusion of these terms adds much complexity to the formulas so they will be neglected.

Expression (9) is first differentiated and powers of ϵ up to and including the fourth are retained. This is then set equal to zero. The exact forms of the coefficients of the various powers of ϵ are quite complex. However, because the discussion is limited to values of A and ϵ which are less than or equal to 0.10, it is possible to approximate these coefficients to an accuracy of two or three percent by very simple expressions. When this is done the following equation results:

$$\epsilon^4+4\theta_0\epsilon^3+4\theta_0 A^2\epsilon-A^4=0, \qquad (10)$$

where**

$$\theta_0=(\varphi^2/Z_0)(1/\omega C_0).$$

The quantity ω is given by $\omega=2\pi f$. Since we can neglect variations in ω during the process of differentiation of (9), the choice of f is somewhat arbitrary. However, its value must be in the neighborhood of the resonant frequency of the free crystal. One convenient choice is the resonant frequency of the free crystal.

The roots of Eq. (10) can be obtained quite readily by factoring. First we regroup terms as follows:

$$(\epsilon^4-A^4)+4\theta_0\epsilon(\epsilon^2-A^2)=0.$$

This can be written as the following product:

$$(\epsilon^2+A^2)(\epsilon^2+4\theta_0\epsilon-A^2)=0, \qquad (11)$$

from which it is immediately clear that there exist only two real roots. These roots which correspond to the resonant and antiresonant frequencies, respectively, are as follows:

$$\epsilon_r=-2\theta_0+(4\theta_0^2+A^2)^{\frac{1}{2}}, \qquad (12a)$$

and

$$\epsilon_{ar}=-2\theta_0-(4\theta_0^2+A^2)^{\frac{1}{2}}. \qquad (12b)$$

When $A=0$, $\epsilon_r=0$ as it must, since $\gamma_c=\pi$ is the condition for evaluating the first resonance of the undamped crystal. Similarly, $\epsilon_{ar}=-4\theta_0$ when $A=0$. If we let Δf be the difference between the antiresonant and the resonant frequencies, and let f_r be the resonant frequency, then we obtain for the ratio $\Delta f/f_r$:

$$\Delta f/f_r=4\theta_0/\pi, \qquad (13a)$$

** This definition of θ_0 is the same as that of θ in CVS (see reference 1).

where

$$\theta_0=(\varphi^2/Z_0)(1/\omega C_0).$$

When the quantity $4\theta_0/\pi$ is expressed in terms of the fundamental constants of the crystal we obtain the expression

$$\Delta f/f_r=[16d^2/\pi s(1+4\pi K-4\pi d^2/s)], \qquad (13b)$$

or in terms of the coupling coefficient*** k,

$$\Delta f/f_r=(4/\pi^2)k^2. \qquad (13c)$$

Upon solving for k we obtain

$$k^2=(\pi^2/4)(\Delta f/f_r). \qquad (13d)$$

Consider now the case in which A is not zero. We can obtain an expression for $\Delta f/f_r$ as follows. From definitions given previously we note:

$$\frac{\Delta f}{f_r}=\frac{f_{ar}-f_r}{f_r}=\frac{\gamma_{ar}-\gamma_r}{\gamma_r}=\frac{(\pi-\epsilon_{ar})-(\pi-\epsilon_r)}{\gamma_r}=\frac{\epsilon_r-\epsilon_{ar}}{\gamma_r}. \qquad (14)$$

The quantity $\epsilon_r-\epsilon_{ar}$ can be obtained by taking the difference between (12a) and (12b). The quantity γ_r in the denominator of (14) is approximately equal to π (within 3 percent).

The following relation is then obtained:

$$\Delta f/f_r=(2/\pi)(4\theta_0^2+A^2)^{\frac{1}{2}}$$

or

$$\Delta f/f_r=(4\theta_0/\pi)(1+(A/2\theta)^2)^{\frac{1}{2}}, \qquad (15)$$

where

$$\theta_0=(\varphi^2/Z_0)(1/\omega C_0).$$

If the coupling coefficient k is introduced into (15), the following form results,

$$\Delta f/f_r=(4/\pi^2)k^2[1+(A/2\theta_0)^2]^{\frac{1}{2}}. \qquad (16a)$$

Expression (16) can be solved for k^2 to yield the expression

$$k^2=(\pi^2/4)(\Delta f/f_r)1/[1+(A/2\theta_0)^2]^{\frac{1}{2}}. \qquad (16b)$$

The effect of damping on the formula for the coupling coefficient is to divide the expression for the undamped case by $[1+(A/2\theta_0)^2]^{\frac{1}{2}}$.

Experimentally, it is generally quite easy to determine the absolute value of the electrical

*** The coupling coefficient is defined by the equation $k=[4\pi d^2/(1+4\pi K-4\pi d^2/s)s]$.

input impedance at resonance and at antiresonance. We therefore define a quality factor, Q', for a crystal system to be the following:

$$Q'=|Z_{ar}|/|Z_r|, \tag{17}$$

where Z_{ar} is the electrical input impedance at an antiresonant frequency and Z_r is the impedance at the corresponding resonant frequency. Each resonant-antiresonant frequency pair might yield a different value for this quality factor. It is therefore necessary to specify which resonant frequency is being considered.

An expression for the quality factor can be obtained by evaluating the electrical input impedance at the resonant frequency and at the antiresonant frequency. Relation (9) can be used with ϵ taking on the two values given by (12a) and (12b). When this is done the following forms result for $|Z_{ar}|$ and $|Z_r|$ after some approximations have been made. These approximations are similar to those used in obtaining the simple forms for the coefficients of Eq. (10).

The expressions for $|Z_r|$ and $|Z_{ar}|$ are:

$$|Z_r|=\frac{1}{\omega C_0}\frac{1}{A\sqrt{2}}\left[1-\frac{1}{(1+(A/2\theta_0)^2)^{\frac{1}{2}}}\right]^{\frac{1}{2}} \tag{18a}$$

and

$$|Z_{ar}|=\frac{1}{\omega C_0}\frac{1}{A\sqrt{2}}\left\{1+\frac{1}{[1+(A/2\theta_0)^2]^{\frac{1}{2}}}\right\}^{\frac{1}{2}}, \tag{18b}$$

where $\omega=2\pi f$ and f can be taken equal to the resonant frequency of the undamped crystal.

The quality factor then follows immediately,

$$Q'=\left\{\frac{[1+(A/2\theta_0)^2]^{\frac{1}{2}}+1}{[1+(A/2\theta_0)^2]^{\frac{1}{2}}-1}\right\}^{\frac{1}{2}}. \tag{19}$$

Experimentally, the quality factor is calculated directly from the measurements. Formula (19) enables us to compute the quantity A if Q' is known. The following expression gives $A/2\theta_0$ in terms of Q',

$$A/2\theta_0=2Q'/Q'^2-1. \tag{20}$$

If $Q'\gg 1$ then

$$A/2\theta_0=2/Q'$$

or

$$A=4\theta_0/Q'. \tag{21}$$

SUMMARY OF RESULTS

The results of this paper can be summarized as follows:

(a) Shift of the first resonant frequency as a result of damping. The resonant frequency is decreased by the amount

$$(1/2\pi L_c/V_c)[-2\theta_0+(4\theta_0^2+A^2)^{\frac{1}{2}}]. \tag{22}$$

(b) The difference between the first resonant frequency and the first antiresonant frequency is given by the following equation:

$$\Delta f/f_r=(4\theta_0/\pi)(1+(A/2\theta_0)^2)^{\frac{1}{2}}. \tag{23}$$

(c) The coupling coefficient is given by

$$k^2=\frac{\pi^2}{4}\frac{\Delta f}{f_r}\frac{1}{[1+(A/2\theta_0)^2]^{\frac{1}{2}}}. \tag{24}$$

(d) The magnitude of the electrical impedance at resonance is

$$|Z_r|=\frac{1}{\omega C_0}\frac{1}{A\sqrt{2}}\left\{1-\frac{1}{[1+(A/2\theta_0)^2]^{\frac{1}{2}}}\right\}^{\frac{1}{2}}. \tag{25}$$

(e) The magnitude of the electrical impedance at antiresonance is

$$|Z_{ar}|=\frac{1}{\omega C_0}\frac{1}{A\sqrt{2}}\left\{1+\frac{1}{[1+(A/2\theta_0)^2]^{\frac{1}{2}}}\right\}^{\frac{1}{2}}. \tag{26}$$

(f) The quality factor Q' is given as follows:

$$Q'=\left\{\frac{[1+(A/2\theta_0)^2]^{\frac{1}{2}}+1}{[1+(A/2\theta_0)^2]^{\frac{1}{2}}-1}\right\}^{\frac{1}{2}}. \tag{27}$$

(g) The quantity A, equal to $(Z_1/Z_0+Z_2/Z_0)$, is given by the equation

$$A/2\theta_0=2Q'/Q'^2-1. \tag{28}$$

If $Q'\gg 1$,

$$A=4\theta_0/Q', \tag{28a}$$

or

$$Q'=4\theta_0/A. \tag{29}$$

In the above formulas $\theta_0=(\varphi^2/Z_0)(1/\omega C_0)$.

The above formulas, (a) to (g), relate various quantities of interest for a vibrating crystal under low loss loading conditions, to the constants of the crystal and the values of the load impedances. The formulas are applicable when the loads are purely resistive. It should be noted that the capacity C_0 appears in the expression for θ_0. In making experimental measurements on the crystal system it is necessary to take into consideration any capacity in parallel with the capacity of the crystal. Under such circumstances, in the above expression for θ_0 the quantity C_0 should be replaced by C_0 plus any such additional parallel capacity. As can be seen from the above formulas, the quality factor Q' is decreased, and the difference between the antiresonant frequency and the resonant frequency is decreased by added parallel capacity.

APPLICATION OF RESULTS

As an application of the above formulas it might be desirable to obtain the value of the quality factor for a vibrating crystal when radiation is the controlling factor. (Let the acoustic impedance of the load be equal to ρV.)

The quality factor Q' is given by the formula (29),

$$Q' = \frac{4\theta_0}{A}. \tag{30}$$

If we consider a 1″ square x cut quartz crystal 0.120″ thick excited at its first resonant frequency (930 kc) for thickness vibration and radiating into CO_2 gas from both sides, we obtain the value of the quality factor as follows (room temperature is assumed): We first note that

$$\theta_0 = (\varphi^2/Z_0)(1/\omega_0 C), \tag{31}$$

where

$$\varphi = (d/S)(A_c/L_c)$$

and

$$Z_0 = A_c\rho_c V_c.$$

From (3) we obtain

$$A = Z_1/Z_0 + Z_2/Z_0, \tag{32}$$

where

$$Z_1/Z_0 = Z_2/Z_0 = \rho V/\rho_c V_c.$$

When the appropriate numerical values are substituted into the above expressions, the value of Q' as obtained from formula (30) is 170. This, then, is the value of the quality factor if the crystal is radiating under the specified conditions.

It is of interest to note how the value of the quality factor is dependent upon the various constants of the crystal and the driven medium.

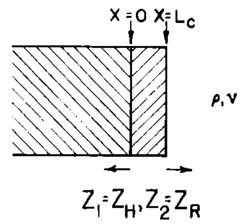

FIG. 2. A crystal fastened in a holder and radiating into a medium of characteristic impedance ρV.

If we combine expressions (30), (31), and (32), and the expression for C_0, i.e.,[3]

$$C_0 = KA_c/4\pi L_c,$$

the following form results:

$$Q' = (4/A)(d/s)^2(4\pi/\rho_c V_c \omega K)(1/L_c), \tag{33}$$

or

$$Q' = (16/A)(d/s)^2(1/\rho_c V_c^2 K).$$

It is readily seen from (33) that the quality factor Q' is independent of the thickness of the crystal and therefore independent of the resonant frequency. The input impedance into the medium driven by the crystal is assumed to be a pure resistance.

The results of this paper make it possible to compute a value for the crystal holder impedance when it can be assumed that this impedance is practically a pure resistance. This calculation is of interest in the analysis of the operation of the double crystal acoustic interferometer which is discussed in an accompanying paper. Consider the arrangement illustrated in Fig. 2.

At $x=0$ the impedance Z_1 is equal to the holder impedance which is assumed to be a pure resistance. At $x=L_c$ the impedance Z_2 is equal to the impedance $Z_R = A_c\rho V$. From an experimental measurement of the quality factor of the crystal in the holder and the use of formula (28) or (28a) it is possible to obtain a value for the holder impedance Z_H. For example, if (28a) is applicable we obtain

$$[Z_H/Z_0 + \rho V/\rho_c V_c] = 4\theta_0/Q' \quad (Q' \gg 1). \tag{34}$$

Upon solving (34) for Z_H/Z_0 the following expression results:

$$Z_H/Z_0 = 4\theta_0/Q' - \rho V/\rho_c V_c, \tag{35}$$

where $Z_0 = A_c\rho_c V_c$. Formula (35) enables us to

[3] See reference 1, p. 153.

obtain reasonably good values for the impedance of the crystal holders of the double crystal acoustic interferometer if we assume that they are practically pure resistances. This assumption is justified by experimental observation of the shift of the resonant frequency of the crystal in the holder as compared with the holder removed. Of course, to apply (35) we must either remove the crystal holders from the interferometer to make the measurements of the quality factor or else introduce some sound absorbing material into the interferometer between the two crystals so that radiation is not reflected back to the crystal.

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The Laplace Transform Solution of Beams

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The application of the Laplace transform to the general problem of the beam is discussed. The method requires no knowledge of normal modes often employed in the classical solution. A solution in closed form is given for the forced vibration of a hinged beam excited by a concentrated harmonic force. The result converges to the correct static curve for zero frequency. The method is extended to the problem of finding the natural frequencies of beams with concentrated mass and spring. The analysis for beams with several masses and springs is outlined.

INTRODUCTION

THE Laplace transform method is often used for transient problems, especially in the fields of electric circuit analysis and heat transmission. Its application to the dynamics of mechanical systems, however, has not been exploited to the same degree, although in many cases it offers attractive advantages over classical methods. For elastic bodies, the forced vibration solution obtained by the transformation also contains the solution for the natural frequencies of free vibration as well as the static deflection curve corresponding to zero frequency.

THE LAPLACE TRANSFORM

The Laplace transform is a functional transformation which in its simplest form is defined by the equation

$$\int_0^\infty f(x)e^{-sx}dx = F(s). \qquad (1)$$

In the above equation the function $f(x)$ is said to be Laplace transformed (abbreviated by $\mathcal{L}$); i.e.,

$$\mathcal{L}[f(x)] = F(s). \qquad (2)$$

When the transformation is applied to the differential equation, it transforms the operation in the real domain x to the complex domain s where the mathematical manipulations become relatively simple. The real solution in x is then obtained by applying an inverse transformation $\mathcal{L}^{-1}$ in the complex field. The method results in a general solution in terms of the one point boundary conditions.

APPLICATION TO THE FORCED VIBRATION OF BEAMS

The general problem of the uniform beam of mass m per unit length excited by a harmonic force $f(x)\sin\omega t$ is expressed by the following partial differential equation. (See Fig. 1.)

$$EI\frac{\partial^4 y}{\partial x^4} = -m\frac{\partial^2 y}{\partial t^2} + f(x)\sin\omega t. \qquad (3)$$

The above equation is readily reduced to an ordinary differential equation of the amplitude by the steady state solution in the form

$$y = w(x)\sin\omega t. \qquad (4)$$

Substituting (4) in (3) we obtain the amplitude

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