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The Double Crystal Acoustic Interferometer*

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A one-dimensional theory of operation of the double crystal acoustic interferometer is presented. The theory is sufficiently general to include both resonance and off resonance operation and any amount of acoustic loading of the crystals. The effects of holder losses and electrical loading of the receiver crystal are also included. A relation is developed which can be used to obtain values of the acoustic velocity and absorption coefficient of the coupling medium from the experimental measurements.

I. INTRODUCTION

NUMEROUS experimental investigations on the acoustic properties of gases and liquids have been carried out with single crystal acoustic interferometers. In such instruments a standing wave system is set up between the crystal and a parallel reflector. In the arrangement commonly used, the change in the acoustic reaction on the crystal as the spacing between reflector and crystal is varied manifests itself as a change in current in some branch of the electric circuit which is used to excite the piezoelectric crystal. Appropriate analysis of the relation between measured current and separation distance yields values of the acoustic velocity and absorption coefficient of the fluid in the interferometer.[1–4] In this analysis the electrical impedance of the crystal is represented by a comparatively simple lumped constant circuit. A change in the acoustic load on the crystal results in a change in value of some of the lumped circuit parameters. This representation is appropriate when the discussion is limited to a restricted band of frequencies about any given frequency. The values of the lumped circuit constants are then functions of the given frequency. Operation in the neighborhood of a resonant frequency of the crystal has been more important experimentally, and has received the major emphasis in the theoretical analyses.

If the procedure of many investigations is followed, it is necessary in order to obtain reasonably accurate values of the acoustic absorption coefficient that the reflector and crystal be very closely parallel, since the shape of one or more of the observed peaks in the current *versus* reflector separation function is necessary to the calculation.[2,5] These shapes are altered drastically if the reflector and crystal are shifted out of parallel by a small amount. Obviously, as the operating frequency is increased, this requirement becomes more exacting.[6]

As indicated in the above references, the theory of the single crystal acoustic interferometer has been extensively discussed in the literature. This paper is concerned with the development of a one-dimensional theory of operation of the double crystal acoustic interferometer. The analysis is based upon formulas derived from the fundamental piezoelectric relations for non-isotropic media.[7] The results obtained are sufficiently general to include both resonance and off resonance operation and any amount of acoustic loading of the crystals by the coupling

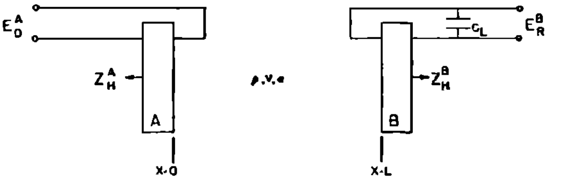

FIG. 1. Schematic diagram of crystal geometry with both electrical and mechanical terminating impedances indicated.

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[1] J. C. Hubbard, Phys. Rev. 38, 1011 (1931).
[2] J. C. Hubbard, Phys. Rev. 41, 523 (1932).
[3] R. S. Alleman, Phys. Rev. 55, 87 (1939).
[4] H. C. Hardy, Ph.D. Thesis, Pennsylvania State College (1941).
[5] W. H. Pielemeier, H. L. Saxton, and D. Telfair, J. Chem. Phys. 8, 106 (1940).
[6] J. L. Stewart, Rev. Sci. Inst. 17, 59 (1946).
[7] See, for example, Appendices C and D of *Design of Crystal Vibrating Systems* by W. J. Fry, J. M. Taylor, and B. W. Henvis (Dover Publications, New York, 1948).

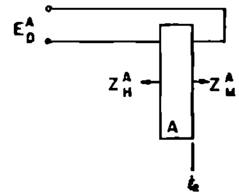

FIG. 2. Driver crystal and terminations.

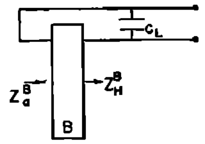

FIG. 3. Receiver crystal and terminations.

medium. The effects of the holder losses and the electrical termination of the receiver crystal are included in the analysis. A simplified set of formulas for the case of light loading, i.e., a gas as the coupling medium, is obtained and applied to a particular experimental arrangement. In the following work it is assumed that the wave-length of the sound is small compared to the dimensions of the crystal face and that the spacing between crystals is not too great, so that diffraction effects can be neglected.[8, 9] Since losses in the interior of the crystal are usually much smaller than losses in the crystal holder and the coupling medium, they have not been included. In addition, it is assumed that crystals are excited in a pure thickness mode of vibration of the piston type.[7] The effect of the "temperature waves" of Herzfeld on the input acoustic impedance into the crystals has not been considered.[10]

The main result of the analysis is an expression which relates the voltage across the receiver crystal to driver voltage, acoustic constants of the coupling medium (velocity and absorption coefficient), losses in the holders, and the electrical termination of the receiver crystal. This formula can be used to determine the acoustic velocity and absorption coefficient from experimental measurements made with the interferometer.

II. GENERAL ANALYSIS**

The problem considered in this paper may be stated as follows. Given two crystals arranged parallel to each other and separated by a distance L (see Fig. 1) to determine the voltage $E_R{}^B$ generated by the receiver crystal B as a function of the following quantities: (a) voltage $E_D{}^A$ across the driver crystal A. (b) mechanical loads $Z_H{}^A$, $Z_H{}^B$ due to the crystal holders on crystal A and B, respectively, (c) electrical load C_L across the receiver crystal, (d) the values of the quantities ρ, V, α characterizing the coupling medium, (e) constants of the crystals.

If the electrical input impedance of the generator which supplies the driving voltage $E_D{}^A$ is small compared to the electrical input impedance of the driving crystal A at all times, then we can assume a constant driving voltage $E_D{}^A$. The analysis of the problem then requires expressions for the following three quantities:

(a) Velocity amplitude of the driver crystal,
(b) acoustic input impedance into the left-hand face of crystal $B(x=L)$,
(c) amplitude of the voltage generated by the receiver crystal B.

We will consider these quantities in the order indicated. In the following the resonant frequencies of a lightly loaded crystal are defined to be the frequencies at which the input electrical impedance characteristic passes through minima. An accompanying paper in this journal on low loss crystal systems should be consulted with regard to the approximation $\gamma_c=\pi$ used in parts of this paper.

(a) Velocity amplitude of the driver crystal

We wish to obtain the velocity amplitude $\dot{\xi}_2$ of the face of the crystal A in contact with the coupling medium, as indicated in Figs. 1 and 2. The mechanical impedance into the holder is indicated by $Z_H{}^A$, the driving voltage by $E_D{}^A$, and the mechanical impedance into the coupling medium by $Z_M{}^A$.

From p. 156 of the *Design of Crystal Vibrating Systems*, hereafter abbreviated CVS, it follows that[7]

$$\dot{\xi}_2=\varphi E_D{}^A\frac{-(\cos\gamma_c+j(Z_H{}^A/Z_0)\sin\gamma_c)(1-\cos\gamma_c)-(Z_H{}^A\cos\gamma_c+jZ_0\sin\gamma_c)((j/Z_0)\sin\gamma_c)}{(\cos\gamma_c+j(Z_H{}^A/Z_0)\sin\gamma_c)(Z_M{}^A)+(Z_H{}^A\cos\gamma_c+jZ_0\sin\gamma_c)}, \quad (1)$$

[8] M. Grabau, J. Acous. Soc. Am. 5, 1 (1933).
[9] E. Grossmann, Physik. Zeits. 35, 83 (1934).
[10] K. F. Herzfeld, Phys. Rev. 53, 899 (1938).
** A table of notation is included in Appendix II.

where

$$\gamma_c = 2\pi f L_c / V_c. \tag{2}$$

Appendix D of CVS indicates the changes in definitions of symbols that are necessary in order to apply the formulas of Part II of CVS to thickness modes. It should be noted that the expressions we use in this paper have been obtained by neglecting certain terms which involve the square of the coupling coefficient. We introduce the shorthand notation

$$\dot{\xi}_2 = E_D{}^A U_\xi(f, Z_M{}^A, Z_H{}^A), \tag{3}$$

where f is the frequency and $Z_M{}^A$ and $Z_H{}^A$ are the mechanical load impedances. $Z_H{}^A$ is the impedance into the holder and $Z_M{}^A$ is the impedance into the bounding medium. Special cases of particular interest for the double crystal acoustic interferometer will now be discussed.

Case 1

The impedance of the holder can be neglected. This means that $Z_H{}^A$ is zero, and the resistive component of the impedance $Z_M{}^A$ is the controlling loss component.

The expression (2) then simplifies to the form

$$\dot{\xi}_2 = \varphi E_D{}^A \frac{-\cos\gamma_c(1-\cos\gamma_c)+\sin^2\gamma_c}{Z_M{}^A \cos\gamma_c + jZ_0 \sin\gamma_c} \quad \text{or} \quad \dot{\xi}_2 = \varphi E_D{}^A \frac{1-\cos\gamma_c}{Z_M{}^A \cos\gamma_c + jZ_0 \sin\gamma_c}. \tag{4}$$

The first resonant frequency of the free crystal is given by $\gamma_c = \pi$. If the load does not appreciably shift this resonant frequency, then the condition $\gamma_c = \pi$ will yield a very good approximation to the resonant frequency of the loaded crystal.

Equation (4) simplifies to

$$\dot{\xi}_2 = -\varphi E_D{}^A (2/Z_M{}^A). \tag{5}$$

Case 2

The holder impedance is purely resistive, that is, $Z_H{}^A$ is a real number. Equation (2) then yields the following:

$$\dot{\xi}_2 = -\varphi E_D{}^A \frac{(\cos\gamma_c + jA\sin\gamma_c)(1-\cos\gamma_c) + (-\sin\gamma_c + A\cos\gamma_c)(\sin\gamma_c)}{(\cos\gamma_c + jA\sin\gamma_c) Z_M{}^A + (A\cos\gamma_c + j\sin\gamma_c) Z_0}, \tag{6}$$

where $A = Z_H{}^A/Z_0$ (a real number). This form reduces to

$$\dot{\xi}_2 = -\varphi E_D{}^A/Z_0 \frac{(\cos\gamma_c + jA\sin\gamma_c) - 1}{(\cos\gamma_c + jA\sin\gamma_c) Z_M{}^A/Z_0 + (A\cos\gamma_c + j\sin\gamma_c)}. \tag{7}$$

The first resonant frequency of the crystal is given very closely by $\gamma_c = \pi$ if the load does not appreciably shift the resonant frequency of the free crystal. If $\gamma_c = \pi$ the expression (7) reduces to

$$\dot{\xi}_2 = (-\varphi E_D{}^A/Z_0)(2/(Z_M{}^A/Z_0) + A). \tag{8}$$

(b) Acoustic Input Impedance into Crystal (see Fig. 3)

We require an expression for the acoustic input impedance into the left face of crystal B. This will involve the mechanical impedance of the holder $Z_H{}^B$, the electrical load across the crystal, and the frequency.

From pages 160 and 161 of CVS we have

$$\frac{Z_M{}^B}{Z_0} = \frac{\theta[2j(1-\cos\gamma_c) + (Z_H{}^B/Z_0)\sin\gamma_c] + (Z_H{}^B/Z_0)\cos\gamma_c + j\sin\gamma_c}{\cos\gamma_c + j(Z_H{}^B/Z_0)\sin\gamma_c + \theta\sin\gamma_c}, \tag{9}$$

where $\theta = [\varphi^2/\omega(C_0 + C_L)Z_0]$.

This formula holds for a pure capacitive load C_L. The quantity Z_H^B is the mechanical impedance of the holder.

The acoustic input impedance Z_a^B can be obtained from (9) by multiplying by $\rho_c V_c$ since $Z_M^B = A_c Z_a^B$ and $Z_0 = A_c \rho_c V_c$.

We thus obtain for the acoustic input impedance into the crystal

$$Z_a^B = \rho_c V_c \frac{\theta[2j(1-\cos\gamma_c)+(Z_H^B/Z_0)\sin\gamma_c]+(Z_H^B/Z_0)\cos\gamma_c+j\sin\gamma_c}{\cos\gamma_c+\theta\sin\gamma_c+j(Z_H^B/Z_0)\sin\gamma_c}, \tag{10}$$

where $\theta = [\varphi^2/\omega(C_0+C_L)Z_0]$.

The electrical load is the capacity C_L, and the mechanical impedance of the holder is Z_H^B.

Two special cases of this formula are of interest.

Case 1

The impedance of the holder can be neglected. This means that the quantity Z_H^B is zero. Expression (10) reduces to the form

$$Z_a^B = j\rho_c V_c \frac{2\theta(1-\cos\gamma_c)+\sin\gamma_c}{\cos\gamma_c+\theta\sin\gamma_c}, \tag{11}$$

or

$$Z_a^B = j\rho_c V_c \frac{2\theta\tan\gamma_c/2+1}{\theta+1/\tan\gamma_c}, \tag{12}$$

where $\theta = [\varphi^2/\omega(C_0+C_L)Z_0]$.

The first resonant frequency as determined from the input electrical impedance characteristic is given closely by $\gamma_c=\pi$ if the mechanical loading is not too great. If $\gamma_c=\pi$,

$$Z_a^B = -j\rho_c V_c 4\theta. \tag{13}$$

Case 2

The holder impedance is a pure resistance that is Z_H^B is a real number. Equation (10) then yields the following:

$$Z_a^B = \rho_c V_c \frac{B[\theta\sin\gamma_c+\cos\gamma_c]+j[2\theta(1-\cos\gamma_c)+\sin\gamma_c]}{[\theta\sin\gamma_c+\cos\gamma_c]+jB\sin\gamma_c}, \tag{14}$$

where $\theta = [\varphi^2/\omega(C_0+C_L)Z_0]$ and $B = Z_H^B/Z_0$.

If the mechanical loads are small the first resonant frequency of the crystal is given closely by $\gamma_c=\pi$. Expression (14) then becomes

$$Z_a^B = \rho_c V_c(B-4j\theta). \tag{15}$$

(c) Voltage Generated by Crystal (see Fig. 4)

An expression for the voltage generated by a crystal as a function of the pressure applied to a face and as a function of the electrical load (the capacity C_L) across the crystal and mechanical load of the holder can be readily obtained (see pages 165–166 of CVS).

This expression is

$$\frac{1}{\delta}\frac{E_R^B}{P^B} = -\frac{Z_L}{Z_e+Z_L}\frac{\tan\gamma_c/2-j(Z_H^B/Z_0)}{2\tan\gamma_c/2-j(Z_H^B/Z_0)}; \tag{16a}$$

where

$$Z_e = \frac{Z_0}{\varphi^2}\frac{(Z_H^B/Z_0)\cos\gamma_c+j\sin\gamma_c}{-2(1-\cos\gamma_c)+j(Z_H^B/Z_0)\sin\gamma_c}, \tag{16b}$$

and

$$Z_L = -[j/\omega(C_0+C_L)]. \tag{16c}$$

As a shorthand notation we let

$$E_R{}^B = -P^B W_E.$$

We consider several special cases of this general expression.

Case 1

The holder impedance is negligible, that is, the quantity $Z_H{}^B$ is zero. For this condition (16) reduces to

$$(1/\delta)(E_R{}^B/P^B) = -\tfrac{1}{2}(Z_L/Z_e+Z_L), \tag{17a}$$

where

$$Z_e = -j(Z_0/2\varphi^2)(1/\tan\gamma_c/2), \tag{17b}$$

and

$$Z_L = -j/\omega(C_0+C_L). \tag{17c}$$

When $\gamma_c = \pi$ Eq. (17b) reduces to $Z_e = 0$.

Case 2

The holder impedance is resistive, i.e., $Z_H{}^B$ is a real number. Let $Z_H{}^B/Z_0 = B$. Then (16) becomes

$$\frac{1}{\delta}\frac{E_R{}^B}{P^B} = -\frac{Z_L}{Z_e+Z_L}\frac{\tan\gamma_c/2 - jB}{2\tan\gamma_c/2 - jB}, \tag{18a}$$

where

$$Z_e = \frac{Z_0}{\varphi^2}\frac{B\cos\gamma_c + j\sin\gamma_c}{-2(1-\cos\gamma_c)+jB\sin\gamma_c}, \tag{18b}$$

and

$$Z_L = -[j/\omega(C_0+C_L)]. \tag{18c}$$

When the loads are not too great the first resonant frequency of the crystal is given closely by $\gamma_c = \pi$. Equations (18) reduce to the following forms when $\gamma_c = \pi$:

$$\frac{1}{\delta}\frac{E_R{}^B}{P^B} = -\frac{1}{2}\frac{Z_L}{Z_e+Z_L}, \tag{19a}$$

$$Z_e = Z_0/\varphi^2 \cdot B/4, \tag{19b}$$

$$Z_L = -[j/\omega(C_0+C_L)]. \tag{19c}$$

Having obtained formulas for the various quantities of interest, we will proceed with the analysis of the double crystal interferometer. Reference should be made to Fig. 1 and Fig. 5. At $x = L$ the acoustic impedance looking to the right is the input acoustic impedance into the crystal B. We designate this by $Z_a{}^B$. Its form is given by the expressions (10), (11), and (14). We can obtain the acoustic input impedance looking to the right at the right-hand face of crystal A by utilizing the expression (3) in Appendix I. The quantity ψ in this expression can be determined from the boundary condition $Z = Z_a{}^B$ at $x = L$, i.e.,

$$Z_a{}^B = \rho V(1/1-j(\alpha/2\pi)) \times \tanh[\psi - (j\omega L/V) - \alpha(L/\lambda)]. \tag{20}$$

The origin is chosen as the reference point so that $a = 0$. We can solve for ψ and obtain

$$\psi = \tanh^{-1}\frac{Z_a{}^B[1-j(\alpha/2\pi)]}{\rho V} + \frac{j\omega L}{V} + \alpha\frac{L}{\lambda}. \tag{21}$$

If we now let $x = 0$ in expression (3) of Appendix I (acoustic impedance) and substitute for ψ the form given in (21), we will obtain the acoustic input impedance into the medium at $x = 0$: i.e., at the right-hand face of crystal A. The result is

$$Z_a(0) = \rho V\frac{1}{1-j\alpha/2\pi}\tanh\left[j\frac{\omega L}{V} + \alpha\frac{L}{\lambda} + \tanh^{-1}\frac{Z_a{}^B(1-j(\alpha/2\pi))}{\rho V}\right]. \tag{22}$$

The mechanical impedance of the load at the right-hand face of crystal A is then

$$Z_M{}^A = A_c Z_a(0), \tag{23}$$

where A_c is the area of the right-hand face of crystal A. The mechanical impedance $Z_M{}^A$ can be expressed as follows:

$$Z_M{}^A = A_c\rho V\frac{1}{1-j\alpha/2\pi}\tanh\left[j\frac{\omega L}{V} + \alpha\frac{L}{\lambda} + \tanh^{-1}\frac{Z_a{}^B[1-j(\alpha/2\pi)]}{\rho V}\right], \tag{24}$$

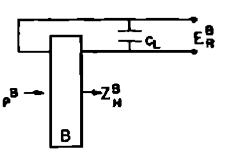

FIG. 4. Receiver crystal with acoustic driving pressure and voltage generated across a capacitative load indicated.

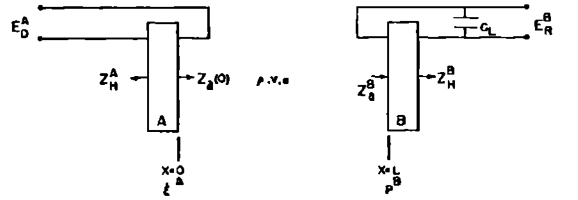

FIG. 5. Schematic diagram of crystals and terminations. The quantities ρ, V, α are the density, acoustic velocity, and absorption coefficient per wave-length of the coupling medium.

or

$$Z_M{}^A = jA_c\rho V\frac{1}{1-j\alpha/2\pi}\tan\left[\frac{\omega L}{V}-j\alpha\frac{L}{\lambda}-j\tanh^{-1}\frac{Z_a{}^B[1-j(\alpha/2\pi)]}{\rho V}\right]. \quad (24a)$$

The velocity amplitude at the face of crystal A at $x=0$ can now be obtained in terms of the driving voltage $E_D{}^A$ and the mechanical impedances of the loads $Z_M{}^A$ and $Z_H{}^A$. Formula (3) yields the required result. In the short notation

$$\dot{\xi}^A = E_D{}^A U_{\dot{\xi}}(f, Z_M{}^A, Z_H{}^A). \quad (25)$$

In order to obtain the voltage across the receiver crystal B it is first necessary to know the pressure amplitude at the face of this crystal at $x=L$. This pressure P^B is readily calculated in terms of the velocity amplitude $\dot{\xi}^A$. We observe that relation (1) of Appendix I on acoustic impedance yields at $x=L$.

$$P^B = 2P_{+a}\times\exp(-\psi)\sinh[\psi-j(\omega L/V)-\alpha(L/\lambda)], \quad (26)$$

where ψ is given by (21) of this section. At $x=0$ relation (2) of the appendix on acoustic impedance yields

$$\dot{\xi}^A = 2P_{+a}\exp(-\psi)(1-j\alpha/2\pi)\cosh\psi/\rho V. \quad (27)$$

The quantity P_{+a} can be eliminated by combining these two equations. We obtain

$$P^B = \dot{\xi}^A\frac{\rho V\sinh[\psi-j(\omega L/V)-\alpha(L/\lambda)]}{(1-j\alpha/2\pi)\cosh\psi}. \quad (28)$$

If we substitute the value of ψ from (21) and perform some algebraic reduction, we obtain

$$P^B = \dot{\xi}^A\frac{Z_a{}^B}{(1-[Z_a{}^B(1-j\alpha/2\pi)/\rho V]^2)^{\frac{1}{2}}}\frac{1}{\cos[(\omega L/V)-j\alpha(L/\lambda)-j\tanh^{-1}[Z_a{}^B(1-j\alpha/2\pi)/\rho V]]}. \quad (29)$$

The voltage amplitude across the receiver crystal can now be expressed in terms of P^B by utilizing (16), (17), or (18) of this section. In short notation

$$E_R{}^B = -P^B W_E. \quad (30)$$

The expressions (25), (29), and (30) can now be combined together to give the voltage amplitude $E_R{}^B$ across the receiver crystal as a function of the driving voltage $E_D{}^A$, the constants of the coupling medium ρ, V, α, the holder impedances $Z_H{}^A$ and $Z_H{}^B$ and the electrical load across the receiver crystal. We obtain

$$E_R{}^B = -\frac{Z_a{}^B}{(1-[Z_a{}^B(1-j\alpha/2\pi)/\rho V]^2)^{\frac{1}{2}}}\frac{W_E U_{\dot{\xi}}}{\cos[(\omega L/V)-j\alpha(L/\lambda)-j\tanh^{-1}Z_a{}^B(1-j\alpha/2\pi)/\rho V]}E_D{}^A. \quad (31)$$

The quantities $U_{\dot{\xi}}$, $Z_a{}^B$, W_E, and $Z_M{}^A$ have been given previously in this section.

Case of Resonance Operation

For the case of maximum receiver voltage for a given driver voltage, the quantity γ_c for the crystals is closely equal to π if a gas is the coupling medium. That is, the crystals are operated very close to their free resonant frequency.

We assume that both crystals have equal first resonant frequencies and that the holder impedances are pure resistances. If γ_c is set equal to π the formula (31) can be written as follows by utilizing (8), (15), (19), and (24a):

$$E_R{}^B = -\frac{Z_a{}^B}{(1-[Z_a{}^B(1-j\alpha/2\pi)/\rho V]^2)^{\frac{1}{2}}}\frac{W_E U_\xi}{\cos[(\omega L/V)-j\alpha(L/\lambda)-j\tanh^{-1}[Z_a{}^B(1-j\alpha/2\pi)/\rho V]]}E_D{}^A, \quad (32)$$

where

$$U_\xi = -(\varphi/Z_0)(2/(Z_M{}^A/Z_0)+A), \quad A = Z_H{}^A/Z_0,$$

$$Z_a{}^B = \rho_c V_c(B-4j\theta), \quad B = Z_H{}^B/Z_0,$$

$$W_E = \delta/2(Z_L/Z_e+Z_L), \quad Z_e = (Z_0/\varphi^2)(B/4), \quad Z_L = -j/\omega(C_0+C_L),$$

$$Z_M{}^A = jA_c\rho V(1/1-j\alpha/2\pi)\tan[(\omega L/V)-j\alpha(L/\lambda)-j\tanh^{-1}[Z_a{}^B(1-j\alpha/2\pi)/\rho V]].$$

The areas of the crystals have been taken equal in these expressions. For a method of determining the quantities A and B see the accompanying paper on low loss crystal systems. It is clear from the form of (32) that for a fixed combination of crystals, electrical load, and coupling medium that the character of the received voltage is completely determined by the function

$$\frac{U_\xi}{\cos[(\omega L/V)-j\alpha(L/\lambda)-j\tanh^{-1}(Z_a{}^B(1-j\alpha/2\pi)/\rho V)]}. \quad (33)$$

The remaining terms in (32) combine together as a single complex constant which determines the amplitude and the phase. However, relative amplitude and phase can be determined from (33).

We will now consider expression (32) in detail. The formula (32) can be written as follows:

$$E_R{}^B = \frac{CU_\xi}{\cos[2\pi(L/\lambda)-j\alpha(L/\lambda)-j\tanh^{-1}(Z_a{}^B(1-j\alpha/2\pi)/\rho V)]}E_D{}^A, \quad (34)$$

$$U_\xi = -(\varphi/Z_0)(2/(Z_M{}^A/Z_0)+A), \quad A = Z_H{}^A/Z_0,$$

$$Z_a{}^B = \rho_c V_c(B-4j\theta), \quad B = Z_H{}^B/Z_0,$$

$$Z_M{}^A = jA_c\rho V(1/1-j\alpha/2\pi)\tan[2\pi(L/\lambda)-j\alpha(L/\lambda)-j\tanh^{-1}(Z_a{}^B(1-j\alpha/2\pi)/\rho V)].$$

If we substitute for U_ξ in (34) and rearrange terms we obtain the following:

$$E_R{}^B = \frac{C'E_D{}^A}{A\cos[2\pi(L/\lambda)-j\alpha(L/\lambda)-j\tanh^{-1}(Z_a{}^B(1-j\alpha/2\pi)/\rho V)] + j(A_c\rho V/Z_0)(1/1-j\alpha/2\pi)\sin[2\pi(L/\lambda)-j\alpha(L/\lambda)-j\tanh^{-1}(Z_a{}^B(1-j\alpha/2\pi)/\rho V)]}. \quad (35)$$

where

$$C' = -(2\varphi/Z_0)C.$$

The quantity $A_c\rho V/Z_0$ reduces to $\rho V/\rho_c V_c$ since $Z_0 = A_c\rho_c V_c$. The relation (35) can also be written in the form

$$E_R{}^B = \frac{C''}{\sin[2\pi(L/\lambda)-j\alpha(L/\lambda)-j\tanh^{-1}(Z_a{}^B(1-j\alpha/2\pi)/\rho V) + \tan^{-1}(A/j(\rho V/\rho_c V_c)(1/1-j\alpha/2\pi))]}E_D{}^A, \quad (36)$$

where

$$C'' = C'/[A^2-(\rho V/\rho_c V_c)^2(1/1-j\alpha/2\pi)^2]^{\frac{1}{2}}.$$

If we insert the expression for Z_a^B into (36) we obtain

$$E_R^B = \frac{C''}{\sin\{2\pi(L/\lambda) - j\alpha(L/\lambda) - j\tanh^{-1}[(\rho_c V_c/\rho V)(B-4j\theta)/(1/1-j\alpha/2\pi)] + \tan^{-1}[(\rho_c V_c/\rho V)(A/j/(1-j\alpha/2\pi))]\}} E_D^A, \quad (37)$$

where

$$\theta = \varphi^2/\omega(C_0 + C_L)Z_0$$

or

$$E_R^B = \frac{C''}{\sin\left\{2\pi\frac{L}{\lambda} - j\alpha\frac{L}{\lambda} + \tan^{-1}\left[\frac{-(\rho_c V_c/\rho V)(1-j\alpha/2\pi)(4\theta + j(A+B))}{1+(\rho_c V_c/\rho V)^2(1-j\alpha/2\pi)^2 A(B-4j\theta)}\right]\right\}} E_D^A. \quad (38)$$

This formula reduces to the following when the spacing distance L is taken equal to zero,

$$E_R^B = \frac{-2\varphi W_E}{A_e} \frac{B-4j\theta}{(A+B)-4j\theta} E_D^A. \quad (39)$$

Note that all quantities involving the constants of the coupling material have vanished from the equation.

Expression (39) can be further simplified to

$$E_R^B = -(Z_L/Z_e + Z_L) \times [(B-4j\theta)/(A+B)-4j\theta] E_D^A. \quad (40)$$

Now if we let

$$\tan^{-1}\frac{-(\rho_c V_c/\rho V)(1-j\alpha/2\pi)(4\theta+j(A+B))}{1+(\rho_c V_c/\rho V)^2(1-j\alpha/2\pi)^2 A(B-4j\theta)} = a - jb, \quad (41)$$

formula (38) can be written

$$E_R^B = \frac{C''}{\sin\{(2\pi(L/\lambda)+a) - j(\alpha(L/\lambda)+b)\}} E_D^A. \quad (42)$$

The absolute value of the ratio of voltage amplitudes, $|E_R^B/E_D^A|$ then becomes

$$|E_R^B/E_D^A| = |C''|/|\sin\{(2\pi(L/\lambda)+a) - j(\alpha(L/\lambda)+b)\}|. \quad (43)$$

This expression can be reduced to the following form by expansion of the sine function

$$|E_R^B/E_D^A| = C''/[\cosh^2(\alpha(L/\lambda)+b) - \cos^2(2\pi(L/\lambda)+a)]^{\frac{1}{2}}. \quad (44)$$

Very convenient upper and lower bounding functions can be obtained for (44) by considering the envelope of the family of curves represented by expression (44) with the quantity "a" as the variable parameter. The quantities α and b are considered fixed in value.

This envelope can be obtained by the ordinary procedure of considering (44) and the expression which results from partial differentiation of (44) with respect to "a" as a set of two simultaneous equations from which "a" is to be eliminated. We proceed as follows. Upon differentiating (44) partially with respect to "a" we obtain

$$\frac{2\pi\cos(2\pi(L/\lambda)+a)\sin(2\pi(L/\lambda)+a)}{[\cosh^2(\alpha(L/\lambda)+b) - \cos^2(2\pi(L/\lambda)+a)]^{\frac{3}{2}}} = 0.$$

This yields

$$\sin 2(2\pi(L/\lambda)+a) = 0.$$

Therefore,

$$2\pi(L/\lambda) + a = (n\pi/2) \quad n = 0, 1, 2\cdots.$$

One envelope function is obtained by choosing even values for n and another is obtained for odd values of n. If n is even the following function results upon elimination of "a" from (44),

$$|E_R^B/E_D^A|_{\text{env}} = |C''|/\sinh(\alpha(L/\lambda)+b). \quad (44\text{a})$$

If n is odd we obtain the following:

$$|E_R^B/E_D^A|_{\text{env}} = |C''|/\cosh(\alpha(L/\lambda)+b). \quad (44\text{b})$$

The characteristics of the formula (44) and the bounding functions (44a) and (44b) will be pointed out with the aid of the illustration presented below.

III. APPLICATION ON THE GENERAL ANALYSIS

Formula (38) will now be applied to the particular system consisting of 1″ square x cut quartz crystals with a first resonant frequency of 930 kc and with CO_2 gas as the coupling medium. Reference should be made to Fig. 5 for a schematic diagram of the arrangement. The quantities A and B can be computed from the formulas given in the accompanying paper on low loss crystal systems.

Formula (35) of that paper can be written as follows, where A has been substituted for the quantity Z_H/Z_0 as indicated previously in this paper;

$$A=(4\theta_0/Q'_A)-(\rho V/\rho_c V_c),\ \text{if}\ Q'_A\gg 1, \quad (45)$$

where Q'_A is the ratio of the impedance at antiresonance (maximum electrical impedance) to the impedance at resonance (minimum electrical impedance). The quantity θ_0 is defined by the expression

$$\theta_0=\varphi^2/Z_0(1/\omega C_0). \quad (46)$$

The results of a computation of the numerical values for the quantities in (38) follow.

1-in. square plate $L_c=0.120$-in. (thickness),
Dielectric constant $=4.5$ (air $=1$),
$C_0=8.34\mu\mu f$ or $8.34(10)^{-12}$ farad,
$\varphi=(d/s)A_c/L_c$
$=0.176(6.43(10)^{-4})/0.305(10)^{-2})$
$=3.71(10)^{-2}$ Coulomb/meter,
$\rho_c V_c=15.2(10)^5$ gram/cm² sec. or $15.2(10)^6$ kg/(meter)² sec.,
$Z_0=A_c\rho_c V_c=9.8(10)^3$ kg/sec.,
$\theta_0=2.89(10)^{-3}$ (dimensionless).

If we choose a value of 60 for the ratio Q' for each crystal, the value of A and B is $1.64(10)^{-4}$. This follows from Eq. (45). The resonant frequency is very closely given by the condition $\gamma_c=\pi$. If the capacity C_L of the load across the detector crystal is $150\mu\mu f$, the value of θ is $1.52(10)^{-4}$.

$$A=B=1.64(10)^{-4},\quad \theta=1.52(10)^{-4},$$

$$(\rho V)_{CO_2}=52.5\ \text{gram/cm}^2\ \text{sec. or}\ 5.25(10)^2\ \text{kg/(meter)}^2\ \text{sec. at } 20°\text{C}.$$

When these values are inserted into (38) the following equation results if the value of α is chosen equal to 0.10.

$$E_R{}^B=\frac{C''}{\sin[2\pi(L/\lambda)-j(0.10)(L/\lambda)-j(0.072)+71.6°]}E_D{}^A. \quad (47)$$

If we collect real and imaginary parts we obtain

$$E_R{}^B=\frac{C''}{\sin[(2\pi(L/\lambda)+71.6°)-j(0.10(L/\lambda)+0.072)]}E_D{}^A. \quad (48)$$

This is in the form of Eq. (42). By observation of (44) the expression (48) yields the following for the ratio of voltage amplitudes,

$$|E_R{}^B/E_D{}^A|=\frac{C''}{[\cosh^2(0.10(L/\lambda)+0.072)-\cos^2(2\pi(L/\lambda)+71.6°)]^{\frac{1}{2}}}.$$

The quantity $(1/|C''|)|E_R{}^B/E_D{}^A|$ is plotted in Fig. 6 as a function of the spacing distance between the crystals in wave-lengths. The bounding functions also indicated on the graph can be obtained from expressions (44a) and (44b).

The value of $|C''|$ can be obtained by utilizing the formulas given above. When these are combined together into one expression the following form results

$$C''=\frac{-2\varphi/Z_0}{(A^2-(\rho V/\rho_c V_c)^2(1/1-j(\alpha/2\pi))^2)^{\frac{1}{2}}}\cdot\frac{Z_a{}^B(\delta/2)}{\{1-[Z_a{}^B(1-j\alpha/2\pi)/\rho V]^2\}^{\frac{1}{2}}}\cdot\frac{-j/\omega(C_0+C_L)}{(Z_0/\varphi^2)(B/4)-j/\omega(C_0+C_L)}. \quad (49)$$

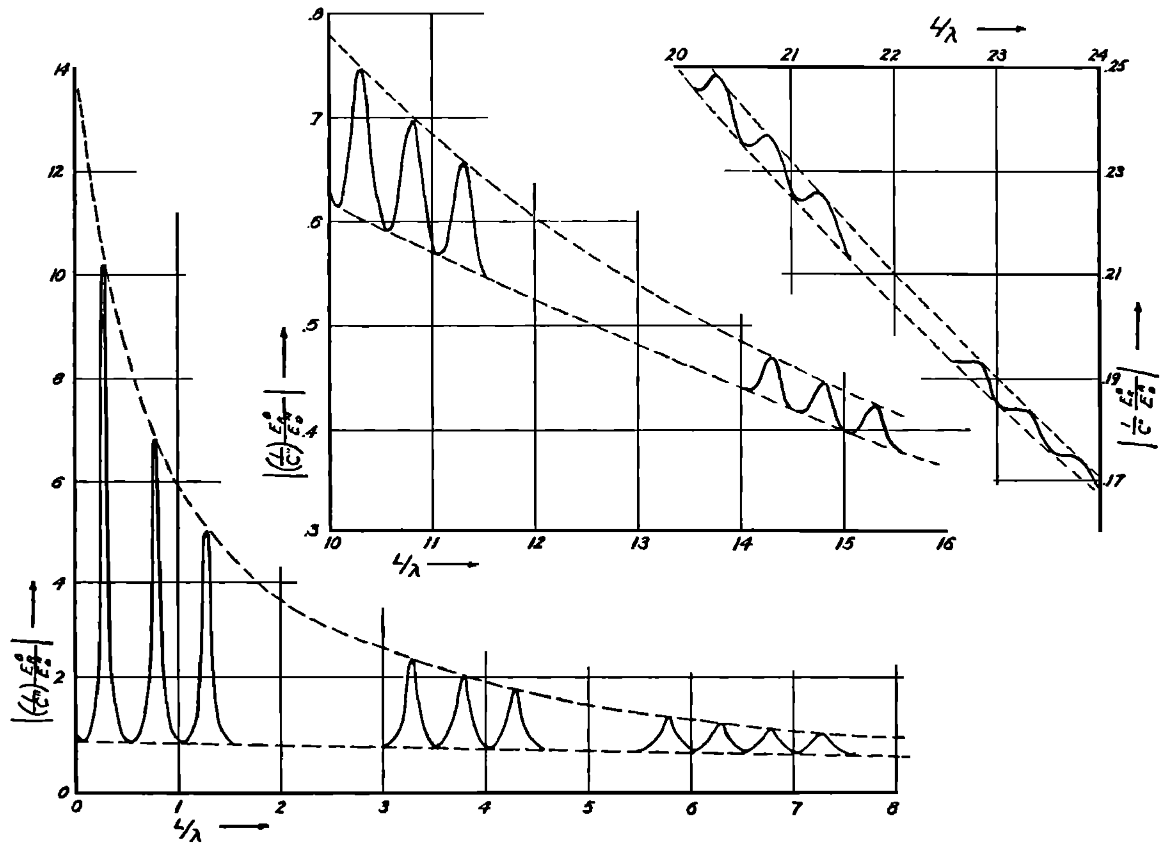

FIG. 6. Typical curve of the ratio of receiver voltage to driver voltage as a function of the crystal spacing in wave-lengths for a highly absorbing medium.

We can obtain the value of C'' upon inserting the values given above. The absolute value is then readily obtained.

IV. DISCUSSION

Some of the general characteristics of the formula (44) and its bounding functions can be discussed and illustrated by means of the graph in Fig. 6. We will first indicate those properties which are characteristic of the general formula. Many other authors have considered this type of function in detail. However, for the sake of completeness a discussion is included here. We are concerned with the following:

(a) Spacing between maxima and minima,
(b) Slope at maxima and minima,
(c) Relative sharpness of peaks and troughs,
(d) Flatness of the function over certain intervals.

Consider first the spacings between maxima and minima. It can be readily seen from the graph in Fig. 6 that when the standing wave ratio is high the maxima and minima fall practically on the bounding functions. However, when the standing wave ratio is small the maxima and minima are shifted off the bounding function. For example, in the region of L/λ values around 23 the graph shows that the shift is of the order of 0.1 wave-length. This means that the maxima and the minima are unequally spaced. The distance between maxima becomes smaller as the spacing distance increases and the distance between minima increases. The distances between the points of tangency with the bounding function is constant and equal to one half wave-length.

It can be readily shown by differentiation of formula (44) and it is also clear from the figure that the slope of the function both at the maximum and at the minimum points is equal to zero.

Concerning the question of the relative sharpness of the peaks and troughs of the function, it can be seen from the figure that when the standing wave ratio is high the peaks are very much sharper than the troughs. As the standing wave ratio decreases the shape of the peaks and the troughs becomes more nearly the same.

It can be seen from the graph that there exists a region of wave-length spacings such that the function is relatively constant or flat over certain intervals in this region. For the case illustrated in the graph this occurs in the neighborhood of wave-length spacings around 23. A region of this type exists for every such function. Its position in spacing distance is determined by the absorption coefficient α.

A few comments which are restricted to the particular case considered in the section on application of the general analysis are now in order. The value 0.10 of the pressure absorption coefficient α is relatively high. It corresponds for example to a CO_2 water vapor mixture excited acoustically in the region of high absorption. The maximum value of the pressure absorption coefficient resulting from vibrational energy transfer in CO_2, H_2O mixtures is about 0.15. From the graph we note that the ratio of the pick-up voltage at the first peak to that at a spacing of 25 to 26λ is about 50.

From an experimental point of view there are various ways to obtain the wave-length and the absorption coefficient from the experimental data. For example, if there is sufficient information to plot a decay curve then the envelopes can be sketched in, and an accurate value of the wave-length can be obtained by determining the distance between two points of contact on the envelope curves. It is clear that the whole curve need not be taken. Only the sections at the two points of contact are needed. Obviously the number of maxima in between must be known.

A fairly good value of the absorption coefficient can be quickly obtained by taking an average of the two envelope curves after they have approached one another fairly closely and noting the rate of decay per wave-length of this average curve. For example, in Fig. 6 if an average of the two envelope curves is taken at 15λ and at 25λ we obtain the pressure absorption coefficient per wave-length as follows. (We assume the average is closely proportional to $e^{-\alpha L/\lambda}$.)

$$0.417 = ce^{-15\alpha}, \quad 0.153 = ce^{-25\alpha}.$$

Dividing these two expressions we obtain

$$e^{10\alpha} = 2.73.$$

Therefore

$$\alpha = 0.10.$$

This checks with the value for which the curve was plotted.

In some experimental arrangements only the differences between the amplitudes of the maxima and minima are known accurately. Appropriate formulas can be derived from the general results of this paper which will enable the observer to obtain values of the absorption coefficient from such data.

SUMMARY OF FORMULAS FOR INTERFEROMETER

The main formulas derived in this paper can be summarized as follows:

The general expression for the voltage across the receiver crystal is

$$E_R{}^B = \frac{Z_a{}^B}{(1-[Z_a{}^B(1-j\alpha/2\pi)/\rho V]^2)^{\frac{1}{2}}} \frac{W_E U_\xi}{\cos\left[\dfrac{\omega L}{V} - j\alpha\dfrac{L}{\lambda} - j\tanh^{-1}(Z_a{}^B(1-j\alpha/2\pi)/\rho V)\right]} E_D{}^A,$$

where W_E is given by either formula (16a), (17a), or (18a), U_ξ is given by either formula (2), (4), or (7), $Z_a{}^B$ is given by either formula (10), (11), or (14).

When the two crystals have equal resonant frequencies and are equal in area and $\gamma_c = \pi$, the voltage across the receiver crystal is

$$E_R{}^B = \frac{C''}{\sin\{(2\pi(L/\lambda)+a) - j(\alpha(L/\lambda)+b)\}} E_D{}^A$$

or in absolute value form

$$\left|\frac{E_R{}^B}{E_D{}^A}\right| = \frac{|C''|}{[\cosh^2(\alpha(L/\lambda)+b)-\cos^2(2\pi(L/\lambda)+a)]^{\frac{1}{2}}},$$

where

$$a-jb=\tan^{-1}\left[\frac{-\rho_c V_c}{\rho V}(1-j\alpha/2\pi)(4\theta+j(A+B))\Big/1+\left(\frac{\rho_c V_c}{\rho V}\right)^2\left(1-j\frac{\alpha}{2\pi}\right)^2 A(B-4j\theta)\right]$$

and

$$C''=\frac{(-2\varphi/Z_0)C}{\left(A^2-\left(\frac{\rho V}{\rho_c V_c}\right)^2\left(\frac{1}{1-j\alpha/2\pi}\right)^2\right)^{\frac{1}{2}}},$$

$$C=\frac{Z_a{}^B}{\left(1-\left[\frac{Z_a{}^B(1-j\alpha/2\pi)}{\rho V}\right]^2\right)^{\frac{1}{2}}}\frac{\delta}{2}\left[\frac{Z_L}{\frac{Z_0 B}{\varphi^2 4}+Z_L}\right].$$

APPENDIX I

Acoustic Impedance Relations

In this appendix we tabulate the expressions for pressure, particle velocity, and acoustic impedance for a medium excited by a plane sound wave.

The relations given here are similar to those given by Morse in his book, *Vibration and Sound*. The differences in sign arise from the use of the sign convention $e^{+j\omega t}$ instead of the $e^{-j\omega t}$ used by Morse. These relations are:

(a) $P=2P_{+a}e^{-\psi}\sinh\left[\psi-\frac{j\omega x}{V}-\alpha\frac{(x-a)}{\lambda}\right]e^{j\omega t},$

(b) $\dot{\xi}=\frac{2P_{+a}}{\rho V}e^{-\psi}\left(1-j\frac{\alpha}{2\pi}\right)\cosh\left[\psi-\frac{j\omega x}{V}-\alpha\frac{(x-a)}{\lambda}\right]e^{j\omega t},$

(c) $Z=\rho V\left(\frac{1}{1-j(\alpha/2\pi)}\right)\tanh\left[\psi-\frac{j\omega x}{V}-\alpha\frac{(x-a)}{\lambda}\right].$

APPENDIX II

Notation

Roman Symbols

- A Ratio of impedances or a sum of such ratios
- A_c Area of crystal
- C_0 Clamped capacity of crystal
- C_L Capacity of electrical load across receiver crystal
- d Piezoelectric constant
- $E_D{}^A$ Voltage across driver crystal
- $E_R{}^B$ Voltage across receiver crystal
- f Frequency
- i Current
- L Spacing distance between crystals
- L_c Thickness of crystal
- P Pressure
- P_+ Pressure amplitude
- Q' Quality factor
- s Elastic constant
- V Sound velocity
- V_c Sound velocity in crystal
- x Space coordinate
- $Z_H{}^A$ Input mechanical impedance into holder of crystal A
- $Z_H{}^B$ Input mechanical impedance into holder of crystal B
- $Z_a{}^B$ Input acoustic impedance into crystal B
- Z_0 $A_c\rho_c V_c$
- $Z_M{}^A$ Mechanical impedance of load on crystal A
- Z_a Electrical input impedance at resonance
- Z_{ar} Electrical input impedance at antiresonance

Greek Symbols

- α Acoustic pressure absorption coefficient per wavelength
- γ_c $\omega L_c/V_c$
- δ $L_c/(d/s)$
- ϵ $\pi-\gamma_c$
- θ_0 $\varphi^2/\omega C_0 Z_0$
- θ $\varphi^2/\omega(C_0+C_L)Z_0$
- φ $d/s(A_c/L_c)$
- ρ density
- ξ particle displacement.